The *Exact Subset Sum with Repetitions* Problem asks: given a collection of positive integers, a target subset sum $S \in \mathbb{N}$, and a subset size $R \in \mathbb{N}$, can we pick exactly R items with repetition from the given collection of integers, that sum to exactly S? More formally,

INSTANCE. Positive integers $\langle a_1, a_2, \ldots, a_m \rangle$, R, and S.

QUESTION. Do there exist non-negative integers $r_1, r_2, \ldots, r_n$ such that

$$\left(\sum_{i=1}^{m} r_i = R \right) \quad \text{and} \quad \left(\sum_{i=1}^{m} r_i \cdot a_i = S \right)? \tag{1}$$

The problem is clearly in NP: given values for the r_i's, we can verify in polynomial time that they satisfy Condition 1. We will show below that it is also NP-hard by reducing from *Exact Cover by 3-Sets*, which is known [1, SP2] to be NP-complete. This problem is defined as follows:

INSTANCE. A set X of $3q$ elements for some $q \in \mathbb{N}$, and a set C of 3-element subsets of X.

QUESTION. Is there a $C' \subseteq C$ such that every element of X is in exactly one of the elements in C'?

The Reduction

Let $X = \{x_o, x_1, \ldots, x_{3q-1}\}$ denote an instance of XC3. Let b_i denote the number in base $(q+1)$ where the i'th digit is one and the rest are zero:

$$b_i = (q+1)^i \text{ for } 0 \le i \le 3q-1$$

We construct an instance of the Exact Subset Sum with Repetitions Problem with the following parameters

$$A = \left\{ \left(\sum_{x_i \in C_j} b_i \right) \;\middle|\; C_j \in C \right\} \tag{2}$$

$$R = q \tag{3}$$

$$S = \sum_{i=0}^{3q-1} b_i \tag{4}$$

Proof of correctness

The target sum S is the base $(q+1)$ number

$$\overbrace{111 \cdots 111}^{3q}{}_{q+1}$$

In order for the i'th digit in this sum to be equal to one, we must include a number from A that includes b_i; this follows from the observation that the $(i-1)$'th digit cannot overflow to the i'th digit since $(q+1)^{i-1}$ is added at most q times, and $q \times (q+1)^{i-1} < (q+1)^i$. □

References

[1] M. Garey and D. Johnson. *Computers and Intractability : A Guide to the Theory of NP-Completeness.* W. H. Freeman and company, NY, 1979.