Counting – Product Rule

– Suppose a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task, then there are $n_1 * n_2$ ways to do the procedure

$$|A \times B| = |A|\ |B|$$

If A and B are finite sets, the number of elements in the Cartesian product of the sets is product of the # elements in each set

Counting – Sum Rule

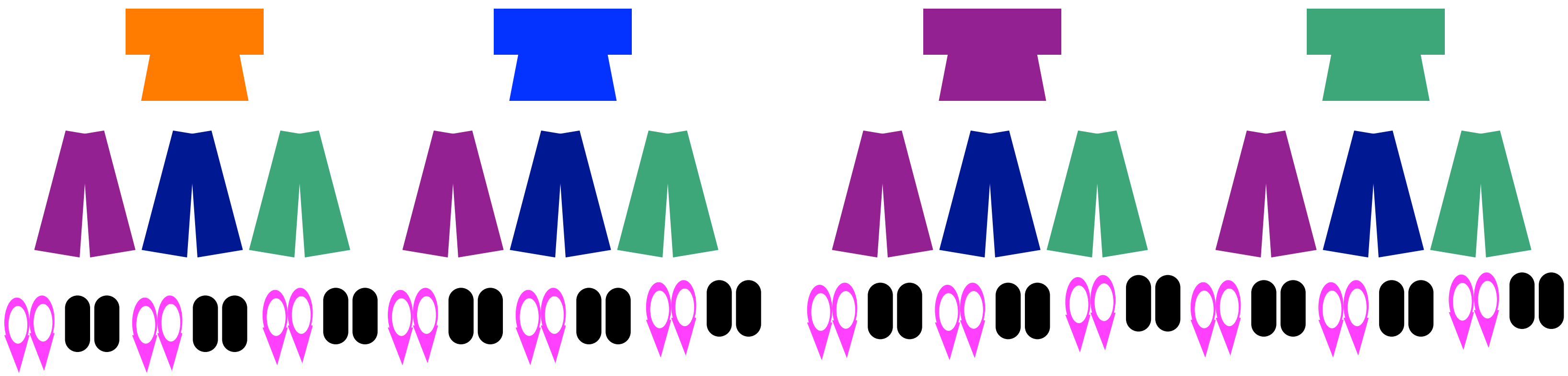
If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task

If A and B are disjoint sets then

$|A \cup B| = |A| + |B|$

Counting

Suppose you have 4 shirts, 3 pairs of pants, and 2 pairs of shoes. How many different outfits do you have?

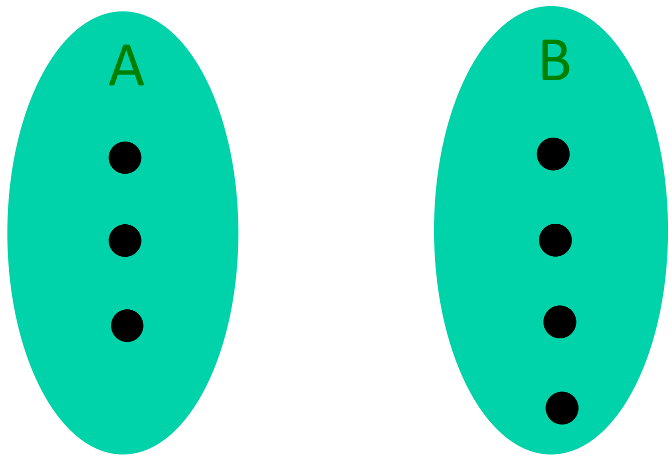


Product Rule

Product Rule

How many functions are there from set A to set B?

64



To define each function we have to make 3 choices, one for each element of A.

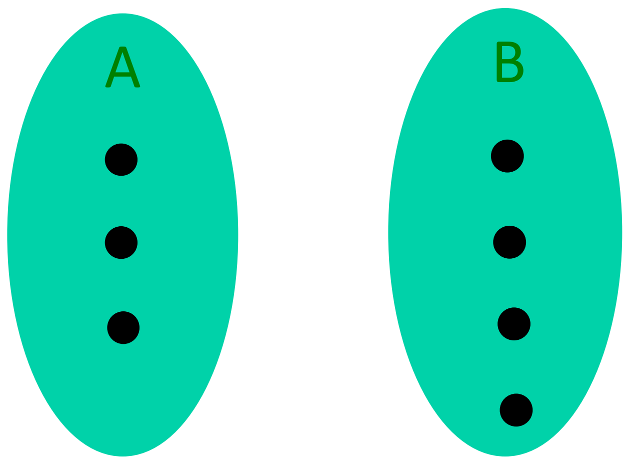
4 4 4

How many ways can each choice be made?

Product Rule

How many one-to-one functions are there from set A to set B?

24



To define each function we have to make 3 choices, one for each element of A.

4 3 2

How many ways can each choice be made?

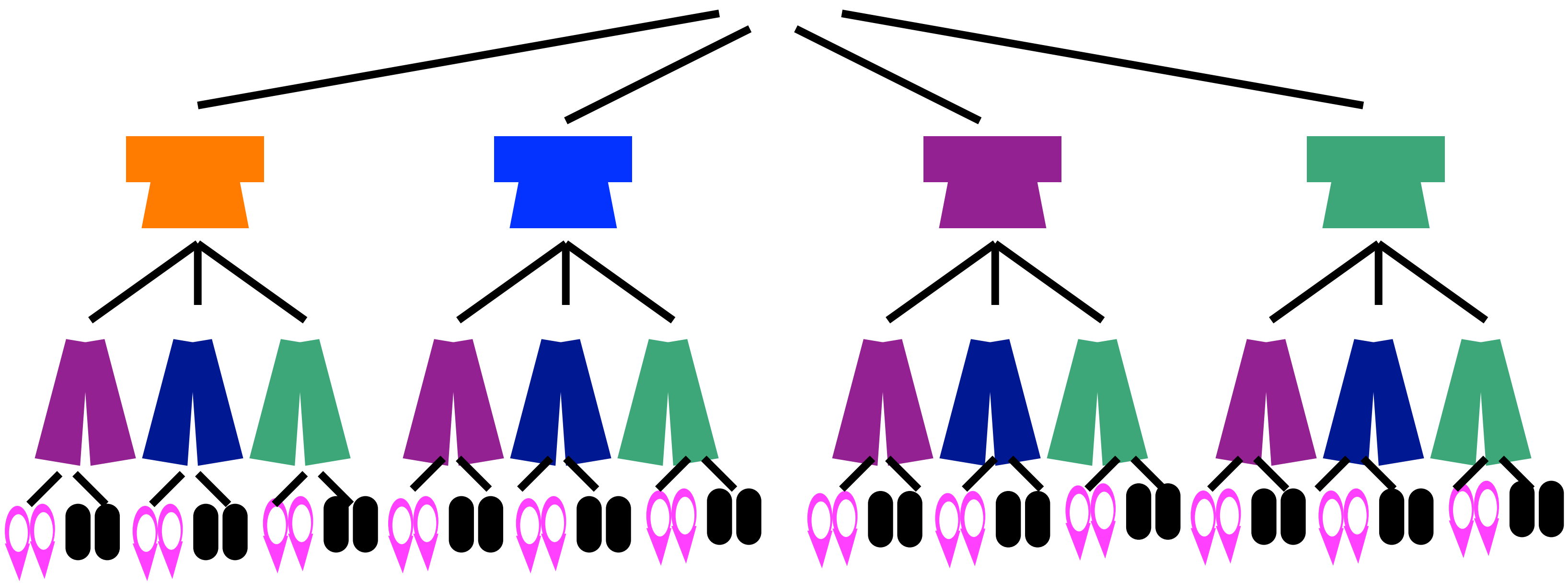
Counting

- **Suppose a student can choose a computer project from one of three lists**
  - List A – 23 projects
  - List B – 15 projects
  - List C – 19 projects
- **No project is on more than one list**
- **How many possible projects are there to choose from?**
  - Just count them up
  - 23 + 15 + 19 = 57

Sum Rule

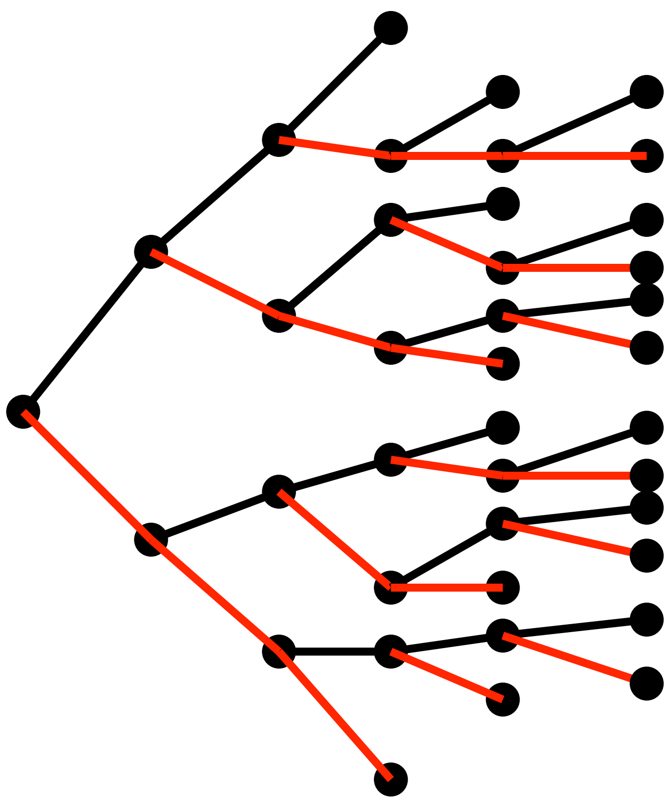
Decision Tree (Tree Diagrams)

Suppose you have 4 shirts, 3 pairs of pants, and 2 pairs of shoes. How many different outfits do you have?



Decision Tree

How many different best of 5 game series were possible between the Cubs and the Dodgers?



20

A DT is a good model for a sequence of events. It assists in counting, and can help you see special structure in the problem.

Inclusion Exclusion Principle

If a task can be done either in one of n_1 ways or in one of n_2 ways, but some of the n_1 ways are the same as some of the n_2 ways.

Are there $n_1 + n_2$ ways to do the task now?

- **If sets A and B are NOT disjoint**
  - $|A \cup B| = |A| + |B| - |A \cap B|$

Example

Count the number of bit strings of length 4 which begin with a 1 or end with 00

How many bit strings of length 4 begin with a 1?

- $2^3 = 8$

How many bit strings end in 00?

- $2^2 = 4$

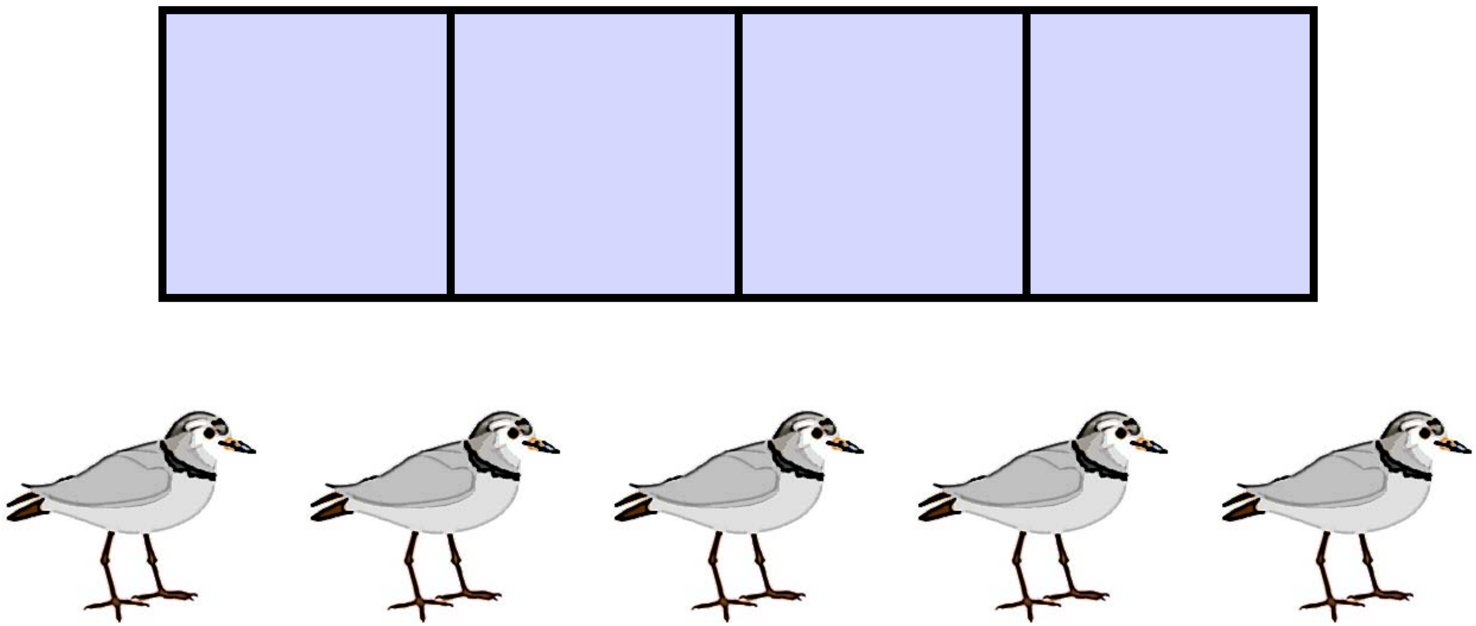
So the total is 8 + 4 = 12 ?

How many bit strings end in 00 and start with 1 ?

- 2

The real total is 8 + 4 – 2 = 10

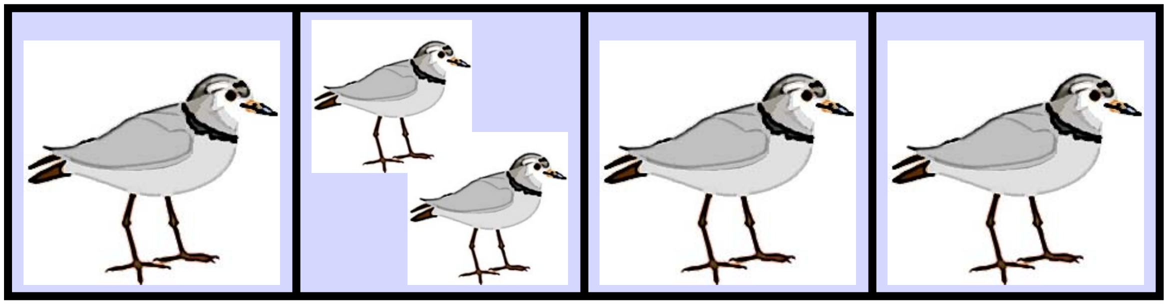
Pigeonhole Principle



If n pigeons fly into k pigeonholes and k < n, then some pigeonhole contains at least two pigeons.

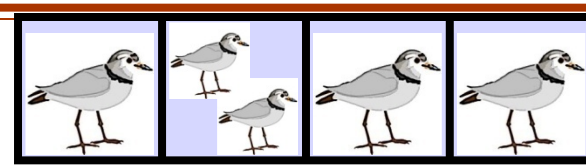
Pigeonhole Principle

We can use this simple little fact to prove amazingly complex things.



If n pigeons fly into k pigeonholes and k < n, then some pigeonhole contains at least two pigeons.

Pigeonhole Principle



Let S contain any 6 positive integers. Then, there is a pair of numbers in S whose difference is divisible by 5.

Let $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Each of these has a remainder when divided by 5. What can these remainders be?

0, 1, 2, 3, or 4

6 numbers, 5 possible remainders…what do we know?

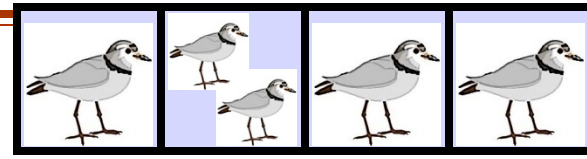
Some pair has the same remainder, by PHP

Consider that pair, a_i and a_j, and their remainder r.

$a_i = 5m + r$, and $a_j = 5n + r$.

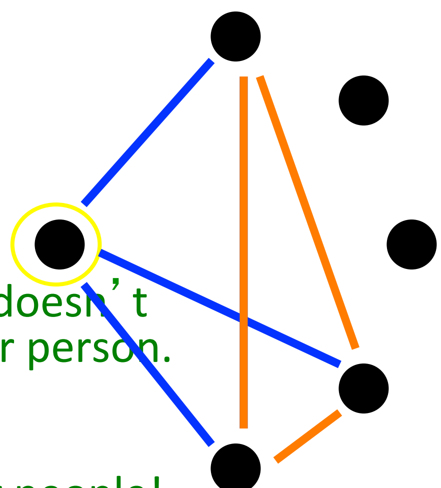
Their difference: $a_i - a_j = (5m + r) - (5n + r) = 5m - 5n = 5(m-n)$, which is divisible by 5.

Pigeonhole Principle



Six people go to a party. Either there is a group of 3 who all know each other, or there is a group of 3 who are all strangers.

Consider one person.



She either knows or doesn't know each other person.

But there are 5 other people! So, she knows, or doesn't know, at least 3 others.

Let's say she knows 3 others.

If any of those 3 know each other, we have a blue △, which means 3 people know each other. If not, then those 3 people must be strangers.

But then we've proven our conjecture for this case.

The case where she *doesn't* know 3 others is similar.

Extra Examples

Extra Examples

In questions below suppose that a “word” is any string of seven letters of the alphabet, with repeated letters allowed.

1) How many words are there?

2) How many words end with the letter T?

3) How many words begin with R and end with T?

4) How many words begin with A or B?

5) How many words begin with A or end with B?

6) How many words begin with A or B and end with A or B?

Extra Examples Solutions

In questions below suppose that a "word" is any string of seven letters of the alphabet, with repeated letters allowed.

1) How many words are there? 26^7

2) How many words end with the letter T? 26^6

3) How many words begin with R and end with T? 26^5

4) How many words begin with A or B? $2 * 26^6$

5) How many words begin with A or end with B?

$26^6 + 26^6 - 26^5$

6) How many words begin with A or B and end with A or B?

$4 * 26^5$

Permutations

A permutation is an ordered arrangement of elements from a set.

On ordered arrangement of r elements of a set is called an
r-permutation

The number of r-permutations of a set with n elements is P(n,r).

If n are r are integers with $0 \leq r \leq n$, then $P(n,r) = \dfrac{n!}{(n-r)!}$

Permutations

In a running race of 12 sprinters, each of the top 5 finishers receives a different medal.

How many ways are there to award the 5 medals?

12 11 10 9 8

a) 60
b) 12^5
c) $12!/7!$
d) 5^{12}
e) No clue

$$P(n,r) = \frac{n!}{(n-r)!}$$

http://www.mathsisfun.com/combinatorics/combinations-permutations-calculator.html

Permutations

Suppose you have time to listen to 10 songs on your daily jog around campus. There are 6 Cold War Kids tunes, 8 Iggy Pop tunes, and 3 Velvet Underground tunes to choose from.

How many different jog playlists can you make?

P(17,10)

Permutations

Suppose you have time to listen to 10 songs on your daily jog around campus. There are 6 Cold War Kids tunes, 8 Iggy Pop tunes, and 3 Velvet Underground tunes to choose from.

Now suppose you want to listen to 4 Cold War Kids, 4 Iggy Pop, and 2 Velvet Underground tunes, in that band order. How many playlists can you make?

P(6,4) x P(8,4) x P(3,2)

Permutations

Suppose you have time to listen to 10 songs on your daily jog around campus. There are 6 Cold War Kids tunes, 8 Iggy Pop tunes, and 3 Velvet Underground tunes to choose from.

Finally, suppose you still want 4 Cold War Kids, 4 Iggy Pop, and 2 Velvet Underground tunes, and the order of the groups does not matter, but you get dizzy and fall down if all the songs by any one group are not played together.

How many playlists are there now?

P(6,4) x P(8,4) x P(3,2) x 3!

Permutations

In how many ways can 5 distinct Martians and 3 distinct Jovians stand in line, if no two Jovians stand together?

___ M1 ___ M2 ___ M3 ___ M4 ___ M5 ___

5! x P(6,3)

Combinations

A combination is an unordered selection of elements from some set.

The number of combinations of r distinct objects chosen from n distinct objects is denoted by C(n,r) or nCr or $\binom{n}{r}$, and is read "n choose r."

$$C(n,r) = P(n,r)/r! = \frac{n!}{((n-r)!r!)}$$

Combinations

A basketball squad consists of 12 players, 5 of which make up a team. How many different teams of players can you make from the 12?

C(12,5)

What's the difference?

In a running race of 12 sprinters, each of the top 5 finishers receives a different medal. How many ways are there to award the 5 medals?

P(12,5)

= C(12,5) x 5!

Combinations

A committee of 3 students is to be selected from a class consisting of 4 freshmen, and 5 sophomores.

In how many ways can a committee with at most 1 freshman be selected?

Start with a easier question first…

What is the total number of possible combinations to select a committee of 3 students ?

Universe = $C(9,3) = 84$

Combinations

Universe = C(9,3) = 84

#Fresh	Comb.	#Soph	Comb.	Both Parts	Total
0	C(4,0)	3	C(5,3)	C(4,0) x C(5,3)	10
1	C(4,1)	2	C(5,2)	C(4,1) x C(5,2)	40
2	C(4,2)	1	C(5,1)	C(4,2) x C(5,1)	30
3	C(4,3)	0	C(5,0)	C(4,3) x C(5,0)	4
				Total Combinations	84

In how many ways can a committee with at most 1 freshman be selected? 10+40 = 50

In how many ways can a committee with at least 1 freshman be selected? 40 + 30 + 4 = 74

Combinations

A committee of 8 students is to be selected from a class consisting of 19 freshmen, and 34 sophomores.

In how many ways can 3 freshmen and 5 sophomores be selected?

C(19,3) x C(34,5)

Combinations

A committee of 8 students is to be selected from a class consisting of 19 freshmen, and 34 sophomores.

In how many ways can a committee with exactly 1 freshman be selected?

C(19,1) x C(34,7)

Combinations

A committee of 8 students is to be selected from a class consisting of 19 freshmen, and 34 sophomores.

In how many ways can a committee with at most 1 freshman be selected?

~~C(34,7) x C(19+27,1)~~

=5379616 * 46
=247,462,336

Break it into cases:

C(34,8) x C(19,0) + C(34,7) x C(19,1) = (18,156,204 x 1) + (5379616 x 19)
= 120,368,908