Announcements

• Homework 5 is due today

• Homework 6 is posted online

• Read Section 2.4 (Sequences and Summations) by Tuesday March 18th
Functions

Suppose we have:

And I ask you to describe the yellow function.

What’s a function?

Notation: \( f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = -(1/2)x - 25 \)

\( f(x) = -(1/2)x - 25 \)
Definition: a function $f : A \rightarrow B$ is a subset of $A \times B$ where $\forall a \in A$, $\exists! b \in B$ and $(a, b) \in f$. 

A collection of points!

A point!
Functions

\[ A = \{\text{Michael, Tito, Janet, Cindy, Bobby}\} \]
\[ B = \{\text{Katherine Jackson, Carol Brady, Mother Teresa}\} \]

Let \( f: A \to B \) be defined as \( f(a) = \text{mother}(a) \).
Functions - image & preimage

For any set $S \subseteq A$, image($S$) = \{b : \exists a \in S, f(a) = b\}

So, image({Michael, Tito}) = ? \quad \text{and} \quad image(A) = ?

-Katherine Jackson
-B - \{Mother Teresa\}

What about the range?
Range is all values the function maps to.

What about the codomain?
Everything in B

image($S$) = f($S$)
Functions - image & preimage

For any \( S \subseteq B \), preimage\((S) = \{ a : \exists b \in S, f(a) = b \} \)

So, preimage\({\{\text{Carol Brady}\}}\) = ? \( \{\text{Cindy, Bobby}\} \)

preimage\( (B) = ? \) \( \text{A} \)

preimage\((S) = f^{-1}(S) \)
A function $f: A \rightarrow B$ is one-to-one (injective, an injection) if $\forall a, b, c,$
$(f(a) = b \land f(c) = b) \rightarrow a = c$

Every $b \in B$ has at most 1 preimage.
A function \( f: A \rightarrow B \) is onto (surjective, or surjection) if \( \forall b \in B, \exists a \in A \ f(a) = b \)

Every \( b \in B \) has at least 1 preimage.
A function \( f: A \rightarrow B \) is bijective if it is one-to-one and onto (also called a one-to-one correspondence).

Every \( b \in B \) has exactly 1 preimage.

An important implication of this characteristic: The preimage \( (f^{-1}) \) is a function!
Functions - examples

Suppose $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+, f(x) = x^2$.

Is $f$ one-to-one? yes
Is $f$ onto? yes
Is $f$ bijective? yes
Suppose \( f: \mathbb{R} \to \mathbb{R}^+, \ f(x) = x^2. \)

Is \( f \) one-to-one? \( \text{yes} \)

Is \( f \) onto? \( \text{no} \)

Is \( f \) bijective? \( \text{no} \)
Functions - examples

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$.

Is $f$ one-to-one? no
Is $f$ onto? no
Is $f$ bijective? no
More Examples

One-to-one, not onto

Not a function

Neither one-to-one nor onto

One-to-one, and onto

Onto, not one-to-one
Functions - composition

Let $f : A \rightarrow B$, and $g : B \rightarrow C$ be functions. Then the composition of $f$ and $g$ is:

$$(g \circ f)(x) = g(f(x))$$
Functions - composition

Let $f_1$ and $f_2$ be functions from the set of integers to the set of integers defined by $f(x) = 2x+3$ and $g(x) = 3x+2$

$$(f \circ g)(x) = f(g(x))$$

$= f(3x+2)$

$= 2(3x +2) +3$

$= 6x + 7$

$$(g \circ f) (x) = g(f(x))$$

$= g(2x + 3)$

$= 3(2x+3) + 2$

$= 6x + 11$
Sequences

**Definition:**
A sequence \( \{a_i\} \) is a function \( f: \mathbb{N} \cup \{0\} \rightarrow \mathbb{R} \), where we write \( a_i \) to indicate \( f(i) \).

**Examples:**

Sequence \( \{a_i\} \), where \( a_i = i \) is just \( a_0 = 0, a_1 = 1, a_2 = 2, \ldots \)

Sequence \( \{a_i\} \), where \( a_i = i^2 \) is just \( a_0 = 0, a_1 = 1, a_2 = 4, \ldots \)
Summation

The symbol: \( \sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k \)

Upper limit \( i=k \)

Lower limit \( i=1 \)
Examples

What is the value of $\sum_{i=1}^{5} i^2$?

$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 1 + 4 + 9 + 16 + 25 = 55$

What if we want the index to run from 0 – 4 instead of 1 – 5?

$\sum_{i=1}^{5} i^2 = \sum_{j=0}^{4} (j + 1)^2$

$= 1 + 4 + 9 + 16 + 25 = 55$
Summation

How do you know this is true?

\[ \sum_{i=1}^{k} (ca_i + b_i) = c \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i \]

• Use associativity to separate the b from the a.

• Use distributivity to factor the c.
Summations you should know...

What is \( S = 1 + 2 + 3 + \ldots + n \)?

\[
S = 1 + 2 + \ldots + n
\]

\[
S = n + n-1 + \ldots + 1
\]

\[
2s = n + 1 + n+1 + \ldots + n+1
\]

- You get \( n \) copies of \( (n+1) \). But we’ve over added by a factor of 2. So just divide by 2.

\[
\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}
\]
Summations you should know...

What is \( S = 1 + 3 + 5 + \ldots + (2n - 1) \)?

\[
\sum_{k=1}^{n} (2k - 1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1
\]

\[
= 2\left(\frac{n(n + 1)}{2}\right) - n
\]

\[
= n^2
\]

Sum of first \( n \) odds.
Summations you should know...

What is $S = 1 + 3 + 5 + \ldots + (2n - 1)$?

$$= n^2$$

Sum of first $n$ odds.
Infinite Cardinality

Two sets $A$ and $B$ have the same cardinality if and only if there exists a bijection (one-to-one correspondence) between them, $A \sim B$.

A set is “countable” if it is either finite or countably infinite.

An infinite set is “countably infinite” if it can be put into “one-to-one correspondence (bijection)” with the set of natural numbers.
Infinite Cardinality

- If there exists a function $f$ from $A$ to $B$ that is injective (i.e. one-to-one) we say that $|A| \leq |B|$

- If there exists a function $f$ from $A$ to $B$ that is surjective (i.e. onto) we say that $|A| \geq |B|$
Set of Java programs countably infinite?

- How would I count them?

- What if I created a Java program generator
  - Program 1 = “a”
  - Program 2 = “b”
  - Program 3 = “c”
  ....

- Eventually you would find programs that compile

- What if I count these programs?
Are rational numbers countable?

• How would I count them?

• What method would I use to list all of the rational numbers?
Infinite Cardinality

Are there more evens than odds?

\{0,2,4,6,8,...\} \sim \{1,3,5,7,9,...\}, f(x) = x-1

Are there more natural numbers than evens?

\mathbb{N} \sim \{0,2,4,6,8,...\}, f(x) = 2x

Are there more evens than multiples of 3?

\{0,2,4,6,8,...\} \sim \{0,3,6,9,12,...\}, f(x) = 3x/2
Infinite Cardinality

How many rational numbers are there?

1/1, 1/2, 1/3, 1/4, ...
2/1, 2/2, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ...

\[ \ldots \]

1/1, 1/2, 2/1, 1/3, 2/2, 3/1, 1/4, 2/3, 3/2, 4/1, \ldots \]
Infinite Cardinality

How many real numbers are in interval [0, 1]?

More irrational between 0 and 1 than all rational everywhere

“Countably many! There’s the list!”

“Are you sure they’re all there?”

Counterexample: 0.536...
So we say the reals are “uncountable.”
Group Problem – Knights and Knaves

There are three people (Alex, Brook and Cody), one of whom is a knight, one a knave, and one a spy.

The knight always tells the truth, the knave always lies, and the spy can either lie or tell the truth.

Alex says: "Cody is a knave."
Brook says: "Alex is a knight."
Cody says: "I am the spy."

Who is the knight, who is the knave, and who is the spy?