Announcements

- Homework 8 is due Today
- Quiz on Tuesday

- Read Sections 9.1 – 9.5 by Tuesday
  - Relations and their properties
  - N-ary relations and their applications
  - Representing relations
  - Closures of relations
  - Equivalence Relations
Chapter 9 - Relations

Recall the definition of the Cartesian (Cross) Product:

The Cartesian Product of sets A and B, $A \times B$, is the set

$$A \times B = \{<x,y> : x \in A \text{ and } y \in B\}.$$  

A relation is just any subset of the CP

$$R \subseteq A \times B$$

- Ex: $A = \text{students}$; $B = \text{courses}$.

  $$R = \{(a,b) \mid \text{student a is enrolled in class b}\}$$
Recall the definition of a function:
\[ f = \{<a,b> : b = f(a), a \in A \text{ and } b \in B\} \]

Is every function a relation? Yes, a function is a special kind of relation.
Properties of Relations

Reflexivity:

A relation $R$ on $A \times A$ is reflexive if for all $a \in A$, $(a,a) \in R$.

Example:

“I am the same height as myself”
Properties of Relations - techniques...

How can we check for the reflexive property?
Draw a picture of the relation (called a “graph”).

- Vertex for every element of A
- Edge for every element of R

Now, what’s R?

\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}

Is this relation reflexive?

Yes!

Loops must exist on EVERY vertex.
Properties of Relations

Symmetry:
A relation $R$ on $A \times A$ is symmetric if $(x,y) \in R$ implies $(y,x) \in R$.

Example:
- Siblings

Anti-symmetry:
A relation on $A \times A$ is anti-symmetric if $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$.

Example:
- Parent/Child

Symmetry and Anti-symmetry are not opposites
A relation can have both or lack both
Most relations cannot be both symmetric and anti-symmetric if it contains some pair of $(a,b)$ where $a \neq b$
Properties of Relations - techniques...

How can we check for the symmetric property?
Draw a picture of the relation (called a “graph”).

Vertex for every element of A
Edge for every element of R

Now, what’s R?
{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

Is this relation symmetric?
No!

EVERY edge must have a return edge.
How can we check for the anti-symmetric property?

Draw a picture of the relation (called a “graph”).

- Vertex for every element of A
- Edge for every element of R

Now, what’s R?

\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}

Is this relation anti-symmetric?

Yes!

No edge can have a return edge.
Transitivity:

A relation on $A \times A$ is transitive if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$.

Example:

Descendents

I am a descendent of my father
My father is a descendent of my grandfather
I am a descendent of my grandfather
Properties of Relations - techniques...

How can we check for transitivity?
Draw a picture of the relation (called a “graph”).

- Vertex for every element of A
- Edge for every element of R

Now, what’s R?
{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

Is this relation transitive?
Yes!
A “short cut” must be present for EVERY path of length 2.
Let $R$ be a relation on people, 
$R=\{(x,y): x \text{ and } y \text{ have lived in the same country}\}$

Is $R$ transitive?  No
Is it symmetric?  Yes
Is it reflexive?  Yes
Is it anti-symmetric?  No
Properties of Relations - techniques...

Let $R$ be a relation on positive integers,
$R=\{(x,y) : 3 \mid (x-y)\}$

Suppose $(x,y)$ and $(y,z)$ are in $R$.
Then we can write $3j = (x-y)$ and $3k = (y-z)$
Can we say $3m = (x-z)$?
Add prev eqn to get: $3j + 3k = (x-y) + (y-z)$
$3(j + k) = (x-z)$

Is $R$ transitive? Yes
Properties of Relations - techniques...

Let $R$ be a relation on positive integers, 
$$R = \{(x,y) : 3 \mid (x-y)\}$$

Does $3k = (x-x)$ for some $k$?  

- Definition of “divides”  
  - Yes, for $k=0$.  

Is $R$ transitive?  
- Yes

Is it reflexive?  
- Yes
Properties of Relations - techniques...

Let R be a relation on positive integers, 
\[ R = \{(x,y): \exists j \in \mathbb{Z}^+ \text{ such that } 3 \mid (x-y) \} \]

Suppose \((x,y)\) is in R.

Then \(3j = (x-y)\) for some \(j\).

Does \(3k = (y-x)\) for some \(k\)?

\[ \text{Definition of “divides”} \]

\[ \text{Yes, for } k = -j. \]

Is R transitive? \(\text{Yes}\)

Is it symmetric? \(\text{Yes}\)

Is it reflexive? \(\text{Yes}\)

Is it anti-symmetric? \(\text{No}\)
More than one relation

Suppose we have 2 relations, $R_1$ and $R_2$, and recall that relations are just sets. So we can take unions, intersections, complements, symmetric differences, etc.

There are other things we can do as well...
Let \( R \) be a relation from \( A \) to \( B \) (\( R \subseteq A \times B \)), and let \( S \) be a relation from \( B \) to \( C \) (\( S \subseteq B \times C \)). The composition of \( R \) and \( S \) is the relation from \( A \) to \( C \) (\( S \circ R \subseteq A \times C \)):

\[
S \circ R = \{(a,c): \exists b \in B, (a,b) \in R, (b,c) \in S\}
\]
More than one relation

Let $R$ be a relation on $A$. Inductively define

$$R^1 = R$$

$$R^{n+1} = R^n \circ R$$

$R^2 = R^1 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1), (4,2)\}$

What is in $R^2$?
More than one relation

Let $R$ be a relation on $A$. Inductively define

$$R^1 = R$$
$$R^{n+1} = R^n \circ R$$

$R^3 = R^2 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2),(4,3)\}$

What is in $R^3$?

... $= R4$

$= R5$

$= R6...$
Practice Problems

• Give an example of the following relations:

• 1) A relation on \{a, b, c\} that is reflexive and transitive, but not antisymmetric

• 2) A relation on \{1, 2\} that is symmetric and transitive, but not reflexive

• 3) A relation on \{1, 2, 3\} that is reflexive and transitive, but not symmetric
Practice Problems (solutions)

• Give an example of the following relations:

• 1) A relation on \{a, b, c\} that is reflexive and transitive, but not antisymmetric
  – \{(a,a), (b,b), (c,c), (a,b), (b,a)\}

• 2) A relation on \{1, 2\} that is symmetric and transitive, but not reflexive
  – \{(1,1)\}

• 3) A relation on \{1, 2, 3\} that is reflexive and transitive, but not symmetric
  – \{(1,1), (2,2), (3,3), (1,2)\}