Announcements

• Quiz 8 is today
• Quiz 9 (Last Quiz) 11/24
• Homework 9 is due on Thursday
• Read 10.1 (Graphs) by Thursday
Exam 2 Grades  (17.5% of your final grade)

Average = 82, Median = 84
(Review) More than one relation

Suppose we have 2 relations, $R_1$ and $R_2$, and recall that relations are just sets. So we can take unions, intersections, complements, symmetric differences, etc.

There are other things we can do as well...
(Review) More than one relation

Let $R$ be a relation from $A$ to $B$ ($R \subseteq A \times B$), and let $S$ be a relation from $B$ to $C$ ($S \subseteq B \times C$). The composition of $R$ and $S$ is the relation from $A$ to $C$ ($S \circ R \subseteq A \times C$):

$$S \circ R = \{(a,c) : \exists b \in B, (a,b) \in R, (b,c) \in S\}$$

$$S \circ R = \{(1,u),(1,v),(2,t),(3,t),(4,u)\}$$
(Review) More than one relation

Let $R$ be a relation on $A$. Inductively define

$$R^1 = R$$

$$R^{n+1} = R^n \circ R$$

$R^2 = R^1 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1), (4,2)\}$
Let $R$ be a relation on $A$. Inductively define

$$R^1 = R$$

$$R^{n+1} = R^n \circ R$$

$$R^3 = R^2 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2),(4,3)\}$$

What is in $R^3$?

... = $R^4$ = $R^5$ = $R^6$...
Relations - A Theorem:

If $R$ is a transitive relation, then $R^n \subseteq R$, $\forall n$.

Was the last example transitive? No

Aside: notice that this theorem allows us to conclude that the previous relation was NOT transitive.

Recall: “if p then q” $\equiv$ “if not q then not p.”

We saw that $R^n$ was not a subset of $R$ (it was growing on every iteration).

Therefore, $R$ is not transitive.
Relations - A Theorem:

If R is a transitive relation, then $R^n \subseteq R$, $\forall n$.

Proof by induction on n.
Base case ($n=1$): $R^1 \subseteq R$ because by definition, $R^1 = R$.
IH: if $R$ is transitive, then $R^n \subseteq R$.
Prove: if $R$ is transitive, then $R^{n+1} \subseteq R$.

We are trying to prove that $R^{n+1} \subseteq R$. To do this, we select an element of $R^{n+1}$ and show that it is also an element of $R$.

Let $(a,b)$ be an element of $R^{n+1}$. Since $R^{n+1} = R^n \circ R$, we know there is an $x$ so that $(a,x) \in R$ and $(x,b) \in R^n$.

By IH, since $R^n \subseteq R$, $(x,b) \in R$.

But wait, if $(a,x) \in R$, and $(x,b) \in R$, and $R$ is transitive, then $(a,b) \in R$. 

Typical way of proving subset.
Consider relation $R=\{(1,2),(2,2),(3,3)\}$ on the set $A = \{1,2,3,4\}$.

Is $R$ reflexive?  

No

What can we add to $R$ to make it reflexive?  

$(1,1), (4,4)$

$R' = R \cup \{(1,1),(4,4)\}$ is called the reflexive closure of $R$. 
Closure

Definition:
The closure of relation $R$ on set $A$ with respect to property $P$ is the relation $R'$ with
1. $R \subseteq R'$
2. $R'$ has property $P$
3. $\forall S \text{ with } R \subseteq S \text{ and } S \text{ has property } P, \ R' \subseteq S.$
4. $R'$ is as small as possible

What does all of this mean?  
P-Closure for a relation might not exist!  
If relation $R$ has property $P$ then $R' = R.$
Reflexive Closure

- Let \( r(R) \) denote the reflexive closure of relation \( R \).
  Then \( r(R) = R \cup \{(a,a) : \forall a \in A\} \)

- Fine, but does that satisfy the definition?
  1. \( R \subseteq r(R) \)
  2. \( r(R) \) is reflexive
  3. Need to show that for any \( S \) with particular properties, \( r(R) \subseteq S \).
     Let \( S \) be such that \( R \subseteq S \) and \( S \) is reflexive. Then
     \( \{(a,a) : \forall a \in A\} \subseteq S \) (since \( S \) is reflexive) and \( R \subseteq S \) (given).
     So, \( r(R) \subseteq S \).

By defn

We added edges!
Symmetric Closure

Let \( s(R) \) denote the symmetric closure of relation \( R \).

Then \( s(R) = R \cup \{(b,a) : (a,b) \in R\} \)

Fine, but does that satisfy the definition?

1. \( R \subseteq s(R) \)
2. \( s(R) \) is symmetric
3. Need to show that for any \( S \) with particular properties, \( s(R) \subseteq S \).

Let \( S \) be such that \( R \subseteq S \) and \( S \) is symmetric. Then \( \{(b,a) : (a,b) \in R\} \subseteq S \) (since \( S \) is symmetric) and \( R \subseteq S \) (given).

So, \( s(R) \subseteq S \).

By defn

We added edges!
Transitive Closure

Let \( c(R) \) denote the transitive closure of relation \( R \).

Then \( c(R) = R \cup \{ (a,c) : \exists b (a,b),(b,c) \in R \} \)

Example: \( A = \{1,2,3,4\} \), \( R = \{(1,2),(2,3),(3,4)\} \).

Apply definition to get:

\[ c(R) = \{(1,2),(2,3),(3,4), (1,3), (2,4)\} \]

Which of the following is true:

a) This set is transitive, but we added too much.
b) This set is the transitive closure of \( R \).
c) This set is not transitive, one pair is missing.
d) This set is not transitive, more than 1 pair is missing.
Transitive Closure

So how DO we find the transitive closure?

Example: A={1,2,3,4}, R={(1,2),(2,3),(3,4)}.

Define a *path* in a relation R, on A to be a sequence of elements from A: a,x₁,...xᵢ,...xₙ₋₁,b, with (a,x₁) ∈ R, ∀i (xᵢ,xᵢ₊₁) ∈ R, (xₙ₋₁,b) ∈ R.

“Path from a to b.”
Transitive Closure

Formally:
If $t(R)$ is the transitive closure of $R$, and if $R$ contains a path from $a$ to $b$, then $(a,b) \in t(R)$

Notes:
- Later book examples will give you efficient algorithms for determining if there is a path between two vertices in a graph (graph connectivity problem)
- Read about Warshall’s algorithm in the text.
Equivalence Relations

Example:
Let $S = \{\text{people in this classroom}\}$, and let

$$R = \{(a,b): \text{a has same # of coins in his/her bag as b}\}$$

Quiz time:
Is $R$ reflexive? Yes
Is $R$ symmetric? Yes
Is $R$ transitive? Yes

This is a special kind of relation, characterized by the properties it has. What’s special about it?

Everyone with the same # of coins as you is just like you.
Equivalence Relations

Formally:
Relation $R$ on $A$ is an *equivalence relation* if $R$ is
Reflexive ($\forall a \in A, aRa$)
Symmetric ($aRb \rightarrow bRa$)
Transitive ($aRb \land bRc \rightarrow aRc$)

"What the heck is $aRb$?"

$aRb$ denotes $(a,b) \in R$.

We can say $a$ is *congruent* to $b$ module $m$, if $m$ divides $a - b$ or $a \equiv b (\text{mod } m)$

Example:
$S = \mathbb{Z}$ (integers), $R = \{(a,b) : a \equiv b \pmod{4}\}$
Is this relation an equivalence relation on $S$?
Have to PROVE reflexive, symmetric, transitive.
Equivalence Relations

Example:

\[ S = \mathbb{Z} \text{ (integers) }, \quad R = \{(a,b) : a \equiv b \mod 4\} \]

Is this relation an equivalence relation on \( S \)?

Start by thinking of \( R \) a different way: \( aRb \) iff there is an int \( k \) so that \( a = 4k + b \). Your quest becomes one of finding \( k \)s.

- Let \( a \) be any integer. \( aRa \) since \( a = 4 \cdot 0 + a \).
- Consider \( aRb \). Then \( a = 4k + b \). But \( b = -4k + a \).
- Consider \( aRb \) and \( bRc \). Then \( a = 4k + b \) and \( b = 4j + c \). So, \( a = 4k + 4j + c = 4(k+j) + c \).
Practice Problems

1) Suppose $R$ and $S$ are relations on \{a,b,c,d\} where 
$R = \{(a,b),(b,c),(c,c),(d,a)\}$ and 
$S = \{(a,c),(b,d),(d,a)\}$

Find the following
A) $R^2$
B) $R^3$
C) $S^2$
D) $R \circ S$
E) $S \circ R$

2) Find the transitive closure of $R$ on \{a,b,c,d\} where 
$R = \{(a,a),(b,a),(b,c),(c,a),(c,c),(c,d),(d,a),(d,c)\}$
Practice Problems Solutions

1) Suppose $R$ and $S$ are relations on $\{a,b,c,d\}$ where $R = \{(a,b),(b,c),(c,c),(d,a)\}$ and $S = \{(a,c),(b,d),(d,a)\}$

Find the following

A) $R^2 = \{(a,c),(b,c),(c,c),(d,b)\}$

B) $R^3 = \{(a,c),(b,c),(c,c),(d,c)\}$

C) $S^2 = \{(b,a),(d,c)\}$

D) $R \circ S = \{(a,c),(b,a),(d,b)\}$

E) $S \circ R = \{(a,d),(d,c)\}$

2) Find the transitive closure of $R$ on $\{a,b,c,d\}$ where $R = \{(a,a),(b,a),(b,c),(c,a),(c,c),(c,d),(d,a),(d,c)\} = R \cup \{(b,d),(d,d)\}$
Quiz