10.3: Graph Representations & Isomorphism

- **Graph representations:**
  - Adjacency lists.
  - Adjacency matrices.
  - Incidence matrices.

- **Graph isomorphism:**
  - Two graphs are isomorphic iff they are identical except for their node names.
Adjacency Lists

A table with 1 row per vertex, listing its adjacent vertices.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Adjacent Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b, c</td>
</tr>
<tr>
<td>b</td>
<td>a, c, e, f</td>
</tr>
<tr>
<td>c</td>
<td>a, b, f</td>
</tr>
<tr>
<td>d</td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>b</td>
</tr>
<tr>
<td>f</td>
<td>c, b</td>
</tr>
</tbody>
</table>
Directed Adjacency Lists

1 row per node, listing the terminal nodes of each edge incident from that node.
## Representing Graphs

### Adjacent Vertices

<table>
<thead>
<tr>
<th>Vertex</th>
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</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b, c, d</td>
</tr>
<tr>
<td>b</td>
<td>a, d</td>
</tr>
<tr>
<td>c</td>
<td>a, d</td>
</tr>
<tr>
<td>d</td>
<td>a, b, c</td>
</tr>
</tbody>
</table>

### Initial Vertex

<table>
<thead>
<tr>
<th>Initial Vertex</th>
<th>Terminal Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>a, b, c</td>
</tr>
</tbody>
</table>
Representing Graphs

Definition: Let $G = (V, E)$ be a simple graph with $|V| = n$. Suppose that the vertices of $G$ are listed in arbitrary order as $v_1, v_2, \ldots, v_n$.

The adjacency matrix $A$ (or $A_G$) of $G$, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with $1$ as its $(i, j)$ entry when $v_i$ and $v_j$ are adjacent, and $0$ otherwise.

In other words, for an adjacency matrix $A = [a_{ij}]$, $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of $G$, $a_{ij} = 0$ otherwise.
Representing Graphs

Example: What is the adjacency matrix $A_G$ for the following graph $G$ based on the order of vertices $a, b, c, d$?

Solution:

$$
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
$$

Note: Adjacency matrices of undirected graphs are always symmetric.
Representing Graphs

**Definition:** Let $G = (V, E)$ be an undirected graph with $|V| = n$. Suppose that the vertices and edges of $G$ are listed in arbitrary order as $v_1, v_2, ..., v_n$ and $e_1, e_2, ..., e_m$, respectively.

The *incidence matrix* of $G$ with respect to this listing of the vertices and edges is the $n \times m$ zero-one matrix with 1 as its $(i, j)$ entry when edge $e_j$ is incident with $v_i$, and 0 otherwise.

In other words, for an incidence matrix $M = [m_{ij}]$, $m_{ij} = 1$ if edge $e_j$ is incident with $v_i$, $m_{ij} = 0$ otherwise.
Representing Graphs

Example: What is the incidence matrix $M$ for the following graph $G$ based on the order of vertices $a$, $b$, $c$, $d$ and edges $1$, $2$, $3$, $4$, $5$, $6$?

Solution:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
a & 1 & 1 & 0 & 0 & 1 & 0 \\
b & 1 & 0 & 1 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 1 & 1 & 1 \\
d & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

Note: Incidence matrices of directed graphs contain two $1$s per column for edges connecting two vertices and one $1$ per column for loops.
Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection (an one-to-one and onto function) $f$ from $V_1$ to $V_2$ with the property that $a$ and $b$ are adjacent in $G_1$ if and only if $f(a)$ and $f(b)$ are adjacent in $G_2$, for all $a$ and $b$ in $V_1$.

Such a function $f$ is called an isomorphism.

In other words, $G_1$ and $G_2$ are isomorphic if their vertices can be ordered in such a way that the adjacency matrices $M_{G_1}$ and $M_{G_2}$ are identical.
Isomorphism of Graphs

From a visual standpoint, $G_1$ and $G_2$ are isomorphic if they can be arranged in such a way that their displays are identical (of course without changing adjacency).

Unfortunately, for two simple graphs, each with $n$ vertices, there are $n!$ possible isomorphisms that we have to check in order to show that these graphs are isomorphic.

However, showing that two graphs are not isomorphic can be easy.
Isomorphism of Graphs

For this purpose we can check invariants, that is, properties that two isomorphic simple graphs must both have.

For example, they must have

- the same number of vertices,
- the same number of edges, and
- the same degrees of vertices.

Note that two graphs that differ in any of these invariants are not isomorphic, but two graphs that match in all of them are not necessarily isomorphic.
Isomorphism of Graphs

Example I: Are the following two graphs isomorphic?

Solution: Yes, they are isomorphic, because they can be arranged to look identical. You can see this if in the right graph you move vertex b to the left of the edge \{a, c\}. Then the isomorphism f from the left to the right graph is: f(a) = E, f(b) = A, f(c) = B, f(d) = C, f(e) = D.
Isomorphism of Graphs

Example II: How about these two graphs?

Solution: No, they are not isomorphic, because they differ in the degrees of their vertices.

Vertex d in right graph is of degree one, but there is no such vertex in the left graph.
10.5: Euler & Hamilton Paths

An **Euler circuit (cycle)** in a graph $G$ is a simple circuit containing every edge of $G$.

An **Euler path** in $G$ is a simple path containing every edge of $G$.

A **Hamilton circuit (cycle)** is a circuit that traverses each vertex in $G$ exactly once.

A **Hamilton path** is a path that traverses each vertex in $G$ exactly once.
Seven Bridges of Königsberg Problem

Can we walk through town, crossing each bridge exactly once and return to the start?

The original problem

Equivalent multigraph
Euler Path Theorems

Theorem: A connected multigraph has an Euler circuit iff each vertex has even degree.

Theorem: A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree.
  – One is the start, the other is the end.
Euler Circuit Algorithm

Begin with any arbitrary node.

Construct a simple path from it until you get back to start.

Repeat for each remaining subgraph, splicing results back into original cycle.
Euler and Hamilton Paths

A pictorial way to motivate the graph theoretic concepts of Eulerian and Hamiltonian paths and circuits is with two puzzles:

The pencil drawing problem

The taxicab problem
Which of the following pictures can be drawn on paper without ever lifting the pencil, without retracing over any segment and returning to where you started?
Pencil Drawing Problem-Euler Paths

Graph Theory: Which of the following graphs has an Euler path?
Pencil Drawing Problem-Euler Paths

Answer: the left but not the right.
Finding Euler Circuits

Q: Why does the following graph have no Euler circuit?
Finding Euler Circuits

A: It contains a vertex of **odd degree**.
Euler Circuits and Paths

• Which of these has an Euler Circuit?
  – $G_1$ (a,e,c,d,e,b,a)

• Euler Path?
  – $G_3$ (a,c,d,e,b,d,a,b)

\[
\begin{align*}
G_1 & : \quad a \rightarrow b \rightarrow e \rightarrow d \rightarrow c \\
G_2 & : \quad a \rightarrow b \rightarrow e \rightarrow d \rightarrow c \\
G_3 & : \quad a \rightarrow d \rightarrow c \rightarrow e \rightarrow b
\end{align*}
\]
Hamilton Paths and Circuits Definition

DEF: A Hamilton path in a graph $G$ is a path which visits every vertex in $G$ exactly once. A Hamilton circuit (or Hamilton cycle) is a cycle which visits every vertex exactly once, except for the first vertex, which is also visited at the end of the cycle.

NOTE: Again, the definition applies both to undirected as well as directed graphs of all types.
Hamilton Paths and Circuits

Dirac’s Theorem: If G is a simple graph with n vertices with n \( \geq 3 \) such that the degree of every vertex in G is at least \( \frac{n}{2} \), then G has a Hamilton circuit.

Ore’s Theorem: If G is a simple graph with n vertices with n \( \geq 3 \) such that \( \deg(u) + \deg(v) \geq n \) for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.
Round-the-World Puzzle

Can we traverse all the vertices of a dodecahedron, visiting each once?`
Hamilton Circuits and Paths

- Which of these has an Hamilton Circuit?
  - $G_1$ (a,b,c,d,e,a)

- Hamilton Path?
  - $G_1$ (a,b,c,d,e)
  - $G_2$ (a,b,c,d)
Practice Problems
Practice Problems

In the questions below give an example or prove that there are none:

1) A simple graph with 6 vertices, whose degrees are 2, 2, 2, 3, 4, 4
2) A simple digraph with indegrees 0, 1, 2, 2 and outdegrees 0, 1, 1, 3
3) A simple digraph with indegrees 0, 1, 2, 4, 5 and outdegrees 0, 3, 3, 3, 3
4) A graph with 7 vertices that has a Hamilton circuit but no Euler circuit.
Practice Problems Solutions

In the questions below give an example or prove that there are none:

1) A simple graph with 6 vertices, whose degrees are 2, 2, 2, 3, 4, 4
   Not possible to have one vertex of odd degree

2) A simple digraph with indegrees 0, 1, 2, 2 and outdegrees 0, 1, 1, 3

3) A simple digraph with indegrees 0, 1, 2, 4, 5 and outdegrees 0, 3, 3, 3, 3
   None, in a simple graph with five vertices there cannot be a vertex with indegree 5

4) A graph with 7 vertices that has a Hamilton circuit but no Euler circuit.
   has a Hamilton circuit.
   $W_6$