Announcements

• Quiz 2 is today

• No Quiz next week

• Exam 1 next week!
A set is an unordered collection of elements.

Some examples:

\{1, 2, 3\} is the set containing “1” and “2” and “3.”
\{1, 1, 2, 3, 3\} = \{1, 2, 3\} since repetition is irrelevant.
\{1, 2, 3\} = \{3, 2, 1\} since sets are unordered.
\{1, 2, 3, …\} is a way we denote an infinite set (in this case, the natural numbers).
∅ = {} is the empty set, or the set containing no elements.

Note: ∅ ≠ \{∅\}
Set Theory - Definitions and notation

\( x \in S \) means “\( x \) is an element of set \( S \).”
\( x \notin S \) means “\( x \) is not an element of set \( S \).”

\( A \subseteq B \) means “\( A \) is a subset of \( B \).”

or, “\( B \) contains \( A \).”
or, “every element of \( A \) is also in \( B \).”
or, \( \forall x \ ((x \in A) \rightarrow (x \in B)) \).
Set Theory - Definitions and notation

A ⊆ B means “A is a subset of B.”

A = B if and only if A and B have exactly the same elements.

iff, A ⊆ B and B ⊆ A
iff, ∀x ((x ∈ A) ↔ (x ∈ B)).

So to show equality of sets A and B, show:

A ⊆ B
B ⊆ A
Set Theory - Definitions and notation

A ⊂ B means “A is a proper subset of B.”

- A ⊆ B, and A ≠ B.
- ∀x ((x ∈ A) → (x ∈ B)) ∧ ¬∀x ((x ∈ B) → (x ∈ A))
- ∀x ((x ∈ A) → (x ∈ B)) ∧ ∃x ((x ∈ B) ∧ ¬(x ∈ A))

![Diagram showing A as a proper subset of B]
Set Theory - Definitions and notation

Quick examples:
\( \{1,2,3\} \subseteq \{1,2,3,4,5\} \)
\( \{1,2,3\} \subset \{1,2,3,4,5\} \)

Is \( \emptyset \subseteq \{1,2,3\} \)?
Yes! \( \forall x \ (x \in \emptyset) \rightarrow (x \in \{1,2,3\}) \) holds, because \( (x \in \emptyset) \) is false.

Is \( \emptyset \in \{1,2,3\} \)?
No!

Is \( \emptyset \subseteq \{\emptyset,1,2,3\} \)?
Yes!

Is \( \emptyset \in \{\emptyset,1,2,3\} \)?
Yes!
Set Theory - Definitions and notation

Quiz time:

Is \( x \subseteq \{x\} \)?
- No

Is \( \{x\} \subseteq \{x\} \)?
- Yes

Is \( \{x\} \in \{x, \{x\}\} \)?
- Yes

Is \( \{x\} \subseteq \{x, \{x\}\} \)?
- Yes

Is \( \{x\} \in \{x\} \)?
- No
Set Theory - Ways to define sets

- Explicitly: \{John, Paul, George, Ringo\}
- Implicitly: \{1,2,3,…\}, or \{2,3,5,7,11,13,17,…\}
- Set builder: \{ x : x is prime \}, \{ x | x is odd \}.
  
  In general \{ x : P(x) is true \}, where P(x) is some description of the set.

: and | are read “such that” or “where”

Ex. Let D(x,y) denote “x is divisible by y.”
Give another name for
\{ x : \forall y ((y > 1) \land (y < x)) \rightarrow \neg D(x,y) \}.

What is this set of numbers?

Primes
Set Theory - Cardinality

If $S$ is finite, then the *cardinality* of $S$, $|S|$, is the number of distinct elements in $S$.

- If $S = \{1,2,3\}$, $|S| = 3$
- If $S = \{3,3,3,3,3\}$, $|S| = 1$
- If $S = \emptyset$, $|S| = 0$
- If $S = \{1, \{1\}, \{1,\{1\}\}\}$, $|S| = 3$
- If $S = \{0,1,2,3,...\}$, $|S|$ is (one kind of) infinity. (more on this later)
Set Theory - Power sets

If S is a set, then the power set of S is
\[ 2^S = \{ x : x \subseteq S \}. \]

\textbf{If } \text{S} \text{ is } \{\text{a}\}, \text{ then } 2^S = \{\emptyset, \{\text{a}\}\}.

\textbf{If } \text{S} \text{ is } \{\text{a}, \text{b}\}, \text{ then } 2^S = \{\emptyset, \{\text{a}\}, \{\text{b}\}, \{\text{a}, \text{b}\}\}.

\textbf{If } \text{S} \text{ is } \emptyset, \text{ then } 2^S = \{\emptyset\}.

\textbf{If } \text{S} \text{ is } \{\emptyset, \{\emptyset\}\}, \text{ then } 2^S = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.

\textbf{Fact: if } \text{S} \text{ is finite, } |P(S)| = 2^{|S|}. \text{ (if } |S| = n, \text{ then } |P(S)| = 2^n)
Set Theory - Cartesian Product

The *Cartesian Product* of two sets $A$ and $B$ is:

$$A \times B = \{ <a,b> : a \in A \land b \in B \}$$

If $A = \{\text{Charlie, Lucy, Linus}\}$, and $B = \{\text{Brown, VanPelt}\}$, then

$$A \times B = \{<\text{Charlie, Brown}>, <\text{Charlie, VanPelt}>, <\text{Lucy, Brown}>, >, <\text{Lucy, VanPelt}>, <\text{Linus, Brown}>, <\text{Linus, VanPelt}>\}$$

$$A_1 \times A_2 \times \ldots \times A_n = \{<a_1, a_2, \ldots, a_n> : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$
Set Theory - Operators

The *union* of two sets $A$ and $B$ is:

$$A \cup B = \{ x : x \in A \lor x \in B \}$$

If $A = \{\text{Charlie, Lucy, Linus}\}$, and
$B = \{\text{Lucy, Desi}\}$, then

$$A \cup B = \{\text{Charlie, Lucy, Linus, Desi}\}$$
Set Theory - Operators

The *intersection* of two sets $A$ and $B$ is:

$$A \cap B = \{ x : x \in A \land x \in B \}$$

If $A = \{\text{Charlie, Lucy, Linus}\}$, and $B = \{\text{Lucy, Desi}\}$, then

$$A \cap B = \{\text{Lucy}\}$$
Another example

If \( A = \{x : x \text{ is a US president}\} \), and
\( B = \{x : x \text{ is deceased}\} \), then

\[ A \cap B = \{x : x \text{ is a deceased US president}\} \]
Set Theory - Operators

One more example

If $A = \{x : x \text{ is a US president}\}$, and $B = \{x : x \text{ is in this room}\}$, then

$$A \cap B = \{x : x \text{ is a US president in this room}\} = \emptyset$$

Sets whose intersection is empty are called *disjoint* sets.
The **complement** of a set $A$ is:

$$\overline{A} = \{ x : x \notin A \}$$

If $A = \{ x : x \text{ is bored} \}$,
then $A = \{ x : x \text{ is not bored} \}$
Set Theory - Operators

The set difference, $A - B$, is (also written $A \setminus B$):

- $A - B = \{ x : x \in A \land x \notin B \}$
- $A - B = A \cap B$
Set Representation

• How could you represent a set on a computer?
  – Bitmap Representation
    • The set of 10 numbers, with 1, 3, and 5 set
      – 1010100000
  – Linked List

  1 -> 3 -> 5

• How would you complement the set?
  – Bitmap?
  – Linked List?
The symmetric difference, $A \oplus B$, is:

$$A \oplus B = \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \}$$

$$= (A - B) \cup (B - A)$$

like “exclusive or”
Set Theory - Operators

\[ A \oplus B = \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \} \]

\[ = (A - B) \cup (B - A) \]

Proof:
\[ \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \} \]
\[ = \{ x : (x \in A - B) \lor (x \in B - A) \} \]
\[ = \{ x : x \in ((A - B) \cup (B - A)) \} \]
\[ = (A - B) \cup (B - A) \]
Set Theory - Famous Identities

Identity

\[ A \cap U = A \]
\[ A \cup \emptyset = A \]

Domination

\[ A \cup U = U \]
\[ A \cap \emptyset = \emptyset \]

Idempotent

\[ A \cup A = A \]
\[ A \cap A = A \]
Set Theory - Famous Identities

- **Excluded Middle** \( A \cup \overline{A} = U \)

- **Uniqueness** \( A \cap \overline{A} = \emptyset \)

- **Double complement** \( A = A \)
Set Theory - Famous Identities

- **Commutativity**
  \[
  A \cup B = B \cup A \\
  A \cap B = B \cap A
  \]

- **Associativity**
  \[
  (A \cup B) \cup C = A \cup (B \cup C) \\
  (A \cap B) \cap C = A \cap (B \cap C)
  \]

- **Distributivity**
  \[
  A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\
  A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
  \]
Set Theory - Famous Identities

- **DeMorgan’s I**
  
  \[(A \cup B) = A \cap B\]

- **DeMorgan’s II**
  
  \[(A \cap B) = A \cup B\]

Hand waving is good for intuition, but we aim for a more formal proof.
Set Theory – 4 Ways to prove identities

• Show that $A \subseteq B$ and that $B \subseteq A$.

• Use a membership table.

• Use previously proven identities.

• Use logical equivalences to prove equivalent set definitions.

Like truth tables

Like $\equiv$

Not hard, a little tedious
Set Theory – 4 Ways to prove identities

Prove that \((A \cup B) = A \cap B\)

1. (\(\subseteq\)) \((x \in A \cup B) \rightarrow (x \notin A \cup B) \rightarrow (x \notin A \text{ and } x \notin B) \rightarrow (x \in A \cap B)\)

2. (\(\supseteq\)) \((x \in A \cap B) \rightarrow (x \notin A \text{ and } x \notin B) \rightarrow (x \notin A \cup B) \rightarrow (x \in A \cup B)\)
Set Theory – 4 Ways to prove identities

Prove that \((A \cup B) = A \cap B\) using a membership table.

0: x is not in the specified set
1: otherwise

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(\overline{A})</th>
<th>(\overline{B})</th>
<th>(A \cap B)</th>
<th>(A \cup B)</th>
<th>(\overline{A \cup B})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tbody>
</table>

Haven’t we seen this before?
Set Theory – 4 Ways to prove identities

Prove that \((A \cup B) = A \cap B\) using logically equivalent set set definitions.

\[(A \cup B) = \{x : \neg(x \in A \lor x \in B)\}\]

\[= \{x : \neg(x \in A) \land \neg(x \in B)\}\]

\[= \{x : (x \in A) \land (x \in B)\}\]

\[= A \cap B\]
Set Theory - Generalized Union

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k > 1, x = ki, k \in \mathbb{N} \} \]

\( A_1 = \{ 2, 3, 4, \ldots \} \)

\( A_2 = \{ 4, 6, 8, \ldots \} \)

\( A_3 = \{ 6, 9, 12, \ldots \} \)
Set Theory - Generalized Union

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k > 1, x = ki, k \in \mathbb{N} \} \]

Then

\[ \bigcup_{i=2}^{\infty} A_i = ? \]

primes

a) Primes  
b) Composites  
c) \( \emptyset \)  
d) \( \mathbb{N} \)  
e) I have no clue.
Set Theory - Generalized Intersection

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k, x = ki, k \in \mathbb{N} \} \]

\( A_1 = \{1,2,3,4,...\} \)

\( A_2 = \{2,4,6,...\} \)

\( A_3 = \{3,6,9,...\} \)
Set Theory - Generalized Intersection

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n
\]

Ex. Let \( U = \mathbb{N} \), and define:

\[
A_i = \{x : \exists k, x = ki, k \in \mathbb{N}\}
\]

Then

\[
\bigcap_{i=1}^{n} A_i = \text{Multiples of } \text{LCM}(1,\ldots,n)
\]
Set Theory - Inclusion/Exclusion

Example:
How many people are wearing a watch?
How many people are wearing sneakers?

How many people are wearing a watch OR sneakers?

\[ |A \cup B| = |A| + |B| - |A \cap B| \]
Example:
There are 83 cs majors.
40 are taking cs240.
31 are taking cs101.
22 are taking both.

How many are taking neither?

\[ 83 - (40 + 31 - 22) = 34 \]
Set Theory - Generalized Inclusion/Exclusion

Suppose we have:

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]
Quiz