Exam 1
1. Assume that the universe for $x$ is all people and the universe for $y$ is the set of all movies. Write the English statement using the following predicates and any needed quantifiers:

- $S(x, y) : x$ saw $y$
- $L(x, y) : x$ liked $y$
- $A(y) : y$ won an award
- $C(y) : y$ is a comedy.

(a) No comedy won an award.
   \textbf{Solution:} $\forall y (C(y) \rightarrow \neg A(y))$

(b) Lois saw *Casablanca*, but didn’t like it.
   \textbf{Solution:} $S(\text{Lois}, \text{Casablanca}) \land \neg L(\text{Lois}, \text{Casablanca})$.

(c) Some people have seen every comedy.
   \textbf{Solution:} $\exists x \forall y [C(y) \rightarrow S(x, y)]$

(d) No one liked every movie he has seen.
   \textbf{Solution:} $\neg \exists x \forall y [S(x, y) \rightarrow L(x, y)]$.

(e) Ben has never seen a movie that won an award.
   \textbf{Solution:} $\neg \exists y [A(y) \land S(Ben, y)]$. 

2. Use a truth table to show that \((p \land q) \rightarrow r\) is not logically equivalent to \((p \rightarrow r) \land (q \rightarrow r)\). Explain in a sentence why your truth table shows that they aren’t logically equivalent.

**Solution:**

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The two propositions are not logical equivalent because they differ in at least one spot in their truth table.
3. Prove or disprove that for any sets $A, B, C$ it is the case that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

**Solution:** True. $A \setminus (B \cap C) = A \cap \overline{B \cap C} = A \cap (\overline{B} \cup \overline{C}) = (A \cap \overline{B}) \cup (A \cap \overline{C}) = (A \setminus B) \cup (A \setminus C)$.

4. Suppose the variable $x$ represents students, $F(x)$ means “$x$ is a freshman”, and $M(x)$ means “$x$ is a math major”. Match the statement in symbols with one of the English statements in this list that is true given that the statement in symbols is true.

- 1. Some Freshmen are math majors
- 2. Every math major is a freshman
- 3. No math major is a freshman

(a) $\neg \exists x (M(x) \land \neg F(x))$ **Solution:** 2

(b) $\neg \forall x (\neg F(x) \lor \neg M(x))$ **Solution:** 1

(c) $\forall x (\neg M(x) \land \neg F(x))$ **Solution:** 3
5. Prove that there is no smallest positive rational number.

**Solution:** Say there is a smallest rational number $x$ and let $x = a/b$. Consider the number $a/(2b)$. This is smaller than $x$, so this contradicts $x$ being the smallest rational number.

6. Consider an unlucky medical student who has to take 90 tests in a year. Tests are only given on the first full week of any month (so students have the remaining weeks each month to study). That is, in any month, there is exactly one week where tests can be administered that month. Prove that there must be a day where the student takes two tests in the year.

**Solution:** There are 12 months in a year. Thus, there are 12 weeks where tests can be take of $7 \cdot 12 = 84$ days. For the sake of contradiction, say there is at most one test on each day, but then the student only takes 84 tests, a contradiction.
7. Prove that \(3n + 5\) is even if and only if \(n\) is odd.

**Solution:** First we prove that if \(3n + 5\) is even then \(n\) is odd. We do this by proving the contrapositive, if \(n\) is even then \(3n+5\) is odd. Assume \(n\) is even, then \(n\) can be written as \(2k\). Then \(3n+5 = 3(2k)+5 = 2(3k+2)+1\), thus \(2n + 5\) is odd.

Now we prove that if \(n\) is odd then \(3n + 5\) is even. Since \(n\) is odd, it can be written as \(n = 2k+1\). Then \(3n + 5 = 3(2k+1)+5 = 6k+8 = 2(3k+4)\), hence \(3n + 5\) is even.
8. Prove that for every positive integer \( n \geq 1 \), 43 evenly divides the following with no remained \( 6^{n+1} + 7^{2n-1} \). [Hint: Use Induction!]

Solution: Consider the base case when \( n = 1 \). In this case \( 6^2 + 7 = 43 \), which is divisible by 43 evenly.

Now assume that 43 evenly divides \( 6^{k+1} + 7^{2k-1} \) for some \( k \geq 1 \). We now prove the case where \( n = k + 1 \). In this case, we have \( 6^{k+2} + 7^{2k+1} = 6^{k+1} \cdot 6 + 7^{2k-1} \cdot 49 = 6(6^{k+1} + 7^{2k-1}) + 43 \cdot 7^{2k-1} \). This first term is divisible by 43 by the inductive hypothesis and the second term is clearly divisible by 43.