1. (Probabilistic Method) Consider a simple graph $G = (V, E)$. We would like to show that there is a ‘cut’ in the graph $G$ such that at least $|E|/2$ edges cross the cut. A cut of a graph is a partition of the vertices $V$ into two sets $S_1$ and $S_2$. The edges cut by the partition are $C = \{ e \mid e = uv \in E \text{ and } v \in S_1, u \in S_2 \}$. In worlds, $C$ are the edges that have one endpoint in $S_1$ and one endpoint in $S_2$. We want to show for any graph $G$ that there exists some cut such that $|C| \geq |E|/2$.

(a) Consider the following randomized procedure. For each vertex $v \in V$ independently and uniformly at random assign $v$ to $S_1$ or $S_2$. What is the expected number of edges cut?

**Solution:** Let $X_e$ be an indicator random variable where $X_e = 1$ if edge $e$ is cut and 0 otherwise. Let $X = \sum_{e \in E} X_e$. The value of $X$ is the number of edges cut. The probability that $X_e = 1$ is 1/2. This is because there are 4 ways to place the endpoints of $e$ into $S_1$ and $S_2$. Each is equally likely and two of the ways results in $e$ being cut. Since $X_2$ is an indicator variable (0 or 1), it is the case that $E[X_e] = 1/2$. Using the linearity of expectation, we have that $ex[X] = E[\sum_{e \in E} X_e] = \sum_{e \in E} E[X_2] = \sum_{e \in E} 1/2 = |E|/2$.

(b) Use the last part and the ideas in the proof of Markov’s inequality to show that there is a non-zero probability that $|C| \geq |E|/2$.

**Solution:** For the sake of contradiction, say there is 0 probability that $|C| \geq |E|/2$. Then the definition of expectation says that $E[X] = \sum_{x=0}^{|E|/2-1} x Pr[X = x] = \sum_{x=1}^{|E|/2-1} x Pr[X = x] \leq \sum_{x=1}^{|E|/2-1} (|E|/2 - 1) Pr[X = x] = (|E|/2 - 1) \sum_{x=1}^{|E|/2-1} Pr[X = x] = |E|/2 - 1$. However, this contradicts the previous part.

(c) Use the probabilistic method to conclude that there must exist a cut in $G$ where $|C| \geq |E|/2$.

**Solution:** The previous part says that there is a non-zero probability that $C = |E|/2$ or more. Thus, the probabilistic method tells use that there must be a cut $C$ such that $|C| \geq |E|/2$. In particular, since there is a non-zero probability of this event occurring, the event must exist.