32. We give direct proofs that (i) implies (ii), that (ii) implies (iii), and that (iii) implies (i). That will suffice. For the first, suppose that \( x = p/q \) where \( p \) and \( q \) are integers with \( q \neq 0 \). Then \( x/2 = p/(2q) \), and this is rational, since \( p \) and \( 2q \) are integers with \( 2q \neq 0 \). For the second, suppose that \( x/2 = p/q \) where \( p \) and \( q \) are integers with \( q \neq 0 \). Then \( x = (2p)/q \), so \( 3x - 1 = (6p)/q - 1 = (6p - q)/q \) and this is rational, since \( 6p - q \) and \( q \) are integers with \( q \neq 0 \). For the last, suppose that \( 3x - 1 = p/q \) where \( p \) and \( q \) are integers with \( q \neq 0 \). Then \( x = (p/q + 1)/3 = (p + q)/(3q) \), and this is rational, since \( p + q \) and \( 3q \) are integers with \( 3q \neq 0 \).

34. No. This line of reasoning shows that if \( \sqrt{2x^2 - 1} = x \), then we must have \( x = 1 \) or \( x = -1 \). These are therefore the only possible solutions, but we have no guarantee that they are solutions, since not all of our steps were reversible (in particular, squaring both sides). Therefore we must substitute these values back into the original equation to determine whether they do indeed satisfy it.

36. The only conditional statements not shown directly are \( p_1 \leftrightarrow p_2 \), \( p_2 \leftrightarrow p_4 \), and \( p_3 \leftrightarrow p_4 \). But these each follow with one or more intermediate steps: \( p_1 \leftrightarrow p_2 \), since \( p_1 \leftrightarrow p_3 \) and \( p_3 \leftrightarrow p_2 \); \( p_2 \leftrightarrow p_4 \), since \( p_2 \leftrightarrow p_1 \) (just established) and \( p_1 \leftrightarrow p_4 \); and \( p_3 \leftrightarrow p_4 \), since \( p_3 \leftrightarrow p_1 \) and \( p_1 \leftrightarrow p_4 \).

38. We must find a number that cannot be written as the sum of the squares of three integers. We claim that 7 is such a number (in fact, it is the smallest such number). The only squares that can be used to contribute to the sum are 0, 1, and 4. We cannot use two 4’s, because their sum exceeds 7. Therefore we can use at most one 4, which means that we must get 3 using just 0’s and 1’s. Clearly three 1’s are required for this, bringing the total number of squares used to four. Thus 7 cannot be written as the sum of three squares.

40. Suppose that we look at the ten groups of integers in three consecutive locations around the circle (first-second-third, second-third-fourth, \ldots, eighth-ninth-tenth, ninth-tenth-first, and tenth-first-second). Since each number from 1 to 10 gets used three times in these groups, the sum of the sums of the ten groups must equal three times the sum of the numbers from 1 to 10, namely 3 \cdot 55 = 165. Therefore the average sum is 165/10 = 16.5. By Exercise 39, at least one of the sums must be greater than or equal to 16.5, and since the sums are whole numbers, this means that at least one of the sums must be greater than or equal to 17.

42. We show that each of these is equivalent to the statement \((v)\) \( n \) is odd, say \( n = 2k + 1 \). Example 1 showed that \((v)\) implies (i), and Example 8 showed that (i) implies (v). For \((v) \rightarrow (ii)\) we see that \( 1 - n = 1 - (2k + 1) = 2(-k) \) is even. Conversely, if \( n \) were even, say \( n = 2m \), then we would have \( 1 - n = 1 - 2m = 2(-m) + 1 \), so \( 1 - n \) would be odd, and this completes the proof by contraposition that \((ii) \rightarrow (v)\). For \((v) \rightarrow (iii)\), we see that \( n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 \) is odd. Conversely, if \( n \) were even, say \( n = 2m \), then we would have \( n^3 = 2(4m^3) \), so \( n^3 \) would be even, and this completes the proof by contraposition that \((iii) \rightarrow (v)\). Finally, for \((v) \rightarrow (iv)\), we see that \( n^2 + 1 = (2k + 1)^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1) \) is even. Conversely, if \( n \) were even, say \( n = 2m \), then we would have \( n^2 + 1 = 2(2m^2) + 1 \), so \( n^2 + 1 \) would be odd, and this completes the proof by contraposition that \((iv) \rightarrow (v)\).