Split the summation into two parts, apply the inductive hypothesis, and do the algebra:

\[
\sum_{j=0}^{k+1} \left( -\frac{1}{2} \right)^j = \sum_{j=0}^{k} \left( -\frac{1}{2} \right)^j + \left( -\frac{1}{2} \right)^{k+1}
\]

\[
= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \frac{(-1)^{k+1}}{2^{k+1}}
\]

\[
= \frac{2^{k+2} + 2(-1)^k}{3 \cdot 2^{k+1}} + \frac{3(-1)^{k+1}}{3 \cdot 2^{k+1}}
\]

\[
= \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}.
\]

For the last step, we used the fact that \(2(-1)^k = -2(-1)^{k+1}\).

14. We proceed by induction. Notice that the letter \(k\) has been used in this problem as the dummy index of summation, so we cannot use it as the variable for the inductive step. We will use \(n\) instead. For the basis step we have \(1 \cdot 2^1 = (1 - 1)2^{1+1} + 2\), which is the true statement \(2 = 2\). We assume the inductive hypothesis, that

\[
\sum_{k=1}^{n} k \cdot 2^k = (n - 1)2^{n+1} + 2,
\]

and try to prove that

\[
\sum_{k=1}^{n+1} k \cdot 2^k = n \cdot 2^{n+2} + 2.
\]

Splitting the left-hand side into its first \(n\) terms followed by its last term and invoking the inductive hypothesis, we have

\[
\sum_{k=1}^{n+1} k \cdot 2^k = \left( \sum_{k=1}^{n} k \cdot 2^k \right) + (n + 1)2^{n+1} = (n - 1)2^{n+1} + 2 + (n + 1)2^{n+1} = 2n \cdot 2^{n+1} + 2 = n \cdot 2^{n+2} + 2,
\]

as desired.

16. The basis step reduces to \(6 = 6\). Assuming the inductive hypothesis we have

\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)
\]

\[
= \frac{k(k + 1)(k + 2)(k + 3)}{4} + (k + 1)(k + 2)(k + 3)
\]

\[
= (k + 1)(k + 2)(k + 3) \left( \frac{k}{4} + 1 \right)
\]

\[
= (k + 1)(k + 2)(k + 3)(k + 4)\).
\]

18. a) Plugging in \(n = 2\), we see that \(P(2)\) is the statement \(2! < 2^2\).

b) Since \(2! = 2\), this is the true statement \(2 < 4\).

c) The inductive hypothesis is the statement that \(k! < k^k\).

d) For the inductive step, we want to show for each \(k \geq 2\) that \(P(k)\) implies \(P(k + 1)\). In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can prove that \((k + 1)! < (k + 1)^{k+1}\).

e) \((k + 1)! = (k + 1)k! < (k + 1)k^k < (k + 1)(k + 1)^k = (k + 1)^{k+1}\)

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer \(n\) greater than 1.

20. The basis step is \(n = 7\), and indeed \(3^7 < 7!\), since \(2187 < 5040\). Assume the statement for \(k\). Then

\[
3^{k+1} = 3 \cdot 3^k < (k + 1) \cdot 3^k < (k + 1) \cdot k! = (k + 1)!, \text{ the statement for } k + 1.
\]