20. One maximal length increasing sequence is 5, 7, 10, 15, 21. One maximal length decreasing sequence is 22, 7, 3. See Exercise 25 for an algorithm.

22. This follows immediately from Theorem 3, with $n = 10$.

24. This problem was on the International Mathematical Olympiad in 2001, a test taken by the six best high school students from each country. Here is a paraphrase of a solution posted on the Web by Steve Olson, author of a book about this competition entitled Count Down. Make a table listing the 21 boys at the top of each column and the 21 girls to the left of each row. This table will contain $21 \cdot 21 = 441$ boxes. In each box write the number of a problem solved by both that girl and that boy. From the given information, each box will contain a number. Each contestant solved at most six problems, so only six different numbers can appear in any given row or column of 21 boxes. Because $5 \cdot 2 = 10$, at least $21 - 10 = 11$ of the boxes in any given row or column must contain problem numbers that appear three or more times in that row. (This is an application of the idea of the pigeonhole principle.) In each row color red all the boxes containing problem numbers that appear at least three times in that row. So each row will have at least 11 red boxes, and therefore there will be at least $11 \cdot 21 = 231$ boxes colored red. Repeat the process with the columns, using the color blue. Because at least 231 boxes are red and 231 are blue, and there are only 441 boxes in all, some of the boxes will be both red and blue. (Here is the second place where the pigeonhole principle is used.) The problem number in a doubly-colored box represents a problem solved by at least three girls and at least three boys.

26. Let the people be $A, B, C, D,$ and $E$. Suppose the following pairs are friends: $A-B, B-C, C-D, D-E,$ and $E-A$. The other five pairs are enemies. In this example, there are no three mutual friends and no three mutual enemies.

28. Let $A$ be one of the people. She must have either 10 friends or 10 enemies, since if there were 9 or fewer of each, then that would account for at most 18 of the 19 other people. Without loss of generality assume that $A$ has 10 friends. By Exercise 27 there are either 4 mutual enemies among these 10 people, or 3 mutual friends. In the former case we have our desired set of 4 mutual enemies; in the latter case, these 3 people together with $A$ form the desired set of 4 mutual friends.

30. This is clear by symmetry, since we can just interchange the notions of friends and enemies.

32. There are 99,999,999 possible positive salaries less than one million dollars, i.e., from $0.01$ to $999,999.99$. By the pigeonhole principle, if there were more than this many people with positive salaries less than one million dollars, then at least two of them must have the same salary.

34. This follows immediately from Theorem 2, with $N = 8,008,278$ and $k = 1,000,001$ (the number of hairs can be anywhere from 0 to a million).

36. Let $K(x)$ be the number of other computers that computer $x$ is connected to. The possible values for $K(x)$ are 1, 2, 3, 4, 5. Since there are 6 computers, the pigeonhole principle guarantees that at least two of the values $K(x)$ are the same, which is what we wanted to prove.

38. This is similar to Example 9. Label the computers $C_1$ through $C_8$, and label the printers $P_1$ through $P_4$. If we connect $C_k$ to $P_k$ for $k = 1, 2, 3, 4$ and connect each of the computers $C_5$ through $C_8$ to all the printers, then we have used a total of $4 + 4 \cdot 4 = 20$ cables. Clearly this is sufficient, because if computers $C_1$ through $C_4$ need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, since they are connected to all the printers. Now we must show that 19 cables are not enough. Since