32. We apply resolution to give the tautology \((p \lor F) \land (\neg p \lor F) \rightarrow (F \lor F)\). The left-hand side is equivalent to \(p \land \neg p\), since \(p \lor F\) is equivalent to \(p\), and \(\neg p \lor F\) is equivalent to \(\neg p\). The right-hand side is equivalent to \(F\). Since the conditional statement is true, and the conclusion is false, it follows that the hypothesis, \(p \land \neg p\), is false, as desired.

34. Let us use the following letters to stand for the relevant propositions: \(d\) for “logic is difficult”; \(s\) for “many students like logic”; and \(e\) for “mathematics is easy.” Then the assumptions are \(d \lor \neg s\) and \(e \rightarrow \neg d\). Note that the first of these is equivalent to \(s \rightarrow d\), since both forms are false if and only if \(s\) is true and \(d\) is false. In addition, let us note that the second assumption is equivalent to its contrapositive, \(d \rightarrow \neg e\). And finally, by combining these two conditional statements, we see that \(s \rightarrow \neg e\) also follows from our assumptions.

a) Here we are asked whether we can conclude that \(s \rightarrow \neg e\). As we noted above, the answer is yes, this conclusion is valid.

b) The question concerns \(\neg e \rightarrow \neg s\). This is equivalent to its contrapositive, \(s \rightarrow e\). That doesn’t seem to follow from our assumptions, so let’s find a case in which the assumptions hold but this conditional statement does not. This conditional statement fails in the case in which \(s\) is true and \(e\) is false. If we take \(d\) to be true as well, then both of our assumptions are true. Therefore this conclusion is not valid.

c) The issue is \(\neg e \lor d\), which is equivalent to the conditional statement \(e \rightarrow d\). This does not follow from our assumptions. If we take \(d\) to be false, \(e\) to be true, and \(s\) to be false, then this proposition is false but our assumptions are true.

d) The issue is \(\neg d \lor \neg e\), which is equivalent to the conditional statement \(d \rightarrow \neg e\). We noted above that this validly follows from our assumptions.

e) This sentence says \(\neg s \rightarrow (\neg e \lor \neg d)\). The only case in which this is false is when \(s\) is false and both \(e\) and \(d\) are true. But in this case, our assumption \(e \rightarrow \neg d\) is also violated. Therefore, in all cases in which the assumptions hold, this statement holds as well, so it is a valid conclusion.

SECTION 1.7 Introduction to Proofs

2. We must show that whenever we have two even integers, their sum is even. Suppose that \(a\) and \(b\) are two even integers. Then there exist integers \(s\) and \(t\) such that \(a = 2s\) and \(b = 2t\). Adding, we obtain \(a + b = 2s + 2t = 2(s + t)\). Since this represents \(a + b\) as \(2\) times the integer \(s + t\), we conclude that \(a + b\) is even, as desired.

4. We must show that whenever we have an even integer, its negative is even. Suppose that \(a\) is an even integer. Then there exists an integer \(s\) such that \(a = 2s\). Its additive inverse is \(-2s\), which by rules of arithmetic and algebra (see Appendix 1) equals \(2(-s)\). Since this is \(2\) times the integer \(-s\), it is even, as desired.

6. An odd number is one of the form \(2n + 1\), where \(n\) is an integer. We are given two odd numbers, say \(2a + 1\) and \(2b + 1\). Their product is \((2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1\). This last expression shows that the product is odd, since it is of the form \(2n + 1\), with \(n = 2ab + a + b\).

8. Let \(n = m^2\). If \(m = 0\), then \(n + 2 = 2\), which is not a perfect square, so we can assume that \(m \geq 1\). The smallest perfect square greater than \(n\) is \((m + 1)^2\), and we have \((m + 1)^2 = m^2 + 2m + 1 = n + 2m + 1 > n + 2 \cdot 1 + 1 > n + 2\). Therefore \(n + 2\) cannot be a perfect square.

10. A rational number is a number that can be written in the form \(x/y\) where \(x\) and \(y\) are integers and \(y \neq 0\). Suppose that we have two rational numbers, say \(a/b\) and \(c/d\). Then their product is, by the usual rules for multiplication of fractions, \((ac)/(bd)\). Note that both the numerator and the denominator are integers, and that \(bd \neq 0\) since \(b\) and \(d\) were both nonzero. Therefore the product is, by definition, a rational number.