form total amounts of the form 5n for all n ≥ 28 using these gift certificates. (In other words, $135 is the largest multiple of $5 that we cannot achieve.)

To prove this by strong induction, let P(n) be the statement that we can form 5n dollars in gift certificates using just 25-dollar and 40-dollar certificates. We want to prove that P(n) is true for all n ≥ 28. From our work above, we know that P(n) is true for n = 28, 29, 30, 31, 32. Assume the inductive hypothesis, that P(j) is true for all j with 28 ≤ j ≤ k, where k is a fixed integer greater than or equal to 32. We want to show that P(k + 1) is true. Because k - 4 ≥ 28, we know that P(k - 4) is true, that is, that we can form 5(k - 4) dollars. Add one more $25-dollar certificate, and we have formed 5(k + 1) dollars, as desired.

10. We claim that it takes exactly n - 1 breaks to separate a bar (or any connected piece of a bar obtained by horizontal or vertical breaks) into n pieces. We use strong induction. If n = 1, this is trivially true (one piece, no breaks). Assume the strong inductive hypothesis, that the statement is true for breaking into k or fewer pieces, and consider the task of obtaining k + 1 pieces. We must show that it takes exactly k breaks. The process must start with a break, leaving two smaller pieces. We can view the rest of the process as breaking one of these pieces into i + 1 pieces and breaking the other piece into k - i pieces, for some i between 0 and k - 1, inclusive. By the inductive hypothesis it will take exactly i breaks to handle the first piece and k - i - 1 breaks to handle the second piece. Therefore the total number of breaks will be 1 + i + (k - i - 1) = k, as desired.

12. The basis step is to note that 1 = 2^0. Notice for subsequent steps that 2 = 2^1, 3 = 2^1 + 2^0, 4 = 2^2, 5 = 2^2 + 2^0, and so on. Indeed this is simply the representation of a number in binary form (base two). Assume the inductive hypothesis, that every positive integer up to k can be written as a sum of distinct powers of 2. We must show that k + 1 can be written as a sum of distinct powers of 2. If k + 1 is odd, then k is even, so 2^0 was not part of the sum for k. Therefore the sum for k + 1 is the same as the sum for k with the extra term 2^0 added. If k + 1 is even, then (k + 1)/2 is a positive integer, so by the inductive hypothesis (k + 1)/2 can be written as a sum of distinct powers of 2. Increasing each exponent by 1 doubles the value and gives us the desired sum for k + 1.

14. We prove this using strong induction. It is clearly true for n = 1, because no splits are performed, so the sum computed is 0, which equals n(n - 1)/2 when n = 1. Assume the strong inductive hypothesis, and suppose that our first splitting is into piles of i stones and n - i stones, where i is a positive integer less than n. This gives a product i(n - i). The rest of the products will be obtained from splitting the piles thus formed, and so by the inductive hypothesis, the sum of the products will be i(i - 1)/2 + (n - i)(n - i - 1)/2. So we must show that

\[ i(n - i) + \frac{i(i - 1)}{2} + \frac{(n - i)(n - i - 1)}{2} = \frac{n(n - 1)}{2} \]

no matter what i is. This follows by elementary algebra, and our proof is complete.

16. We follow the hint to show that there is a winning strategy for the first player in Chomp played on a 2 x n board that starts by removing the rightmost cookie in the bottom row. Note that this leaves a board with n cookies in the top row and n - 1 cookies in the bottom row. It suffices to prove by strong induction on n that a player presented with such a board will lose if his opponent plays properly. We do this by showing how the opponent can return the board to this form following any nonfatal move this player might make. The basis step is n = 1, and in that case only the poisoned cookie remains, so the player loses. Assume the inductive hypothesis (that the statement is true for all smaller values of n). If the player chooses a nonpoisoned cookie in the top row, then that leaves another board with two rows of equal length, so again the opponent chooses the rightmost cookie in the bottom row, and we are back to the hopeless situation, for some board with fewer than n cookies in the top row. If the player chooses the cookie in the m^{th} column from the left in the bottom