and on the other hand
\[(n + 2)H_{n+1} - (n + 1) = (n + 2) \left( H_n + \frac{1}{n+1} \right) - (n + 1)\]
\[= (n + 2)H_n + \frac{n + 2}{n+1} - (n + 1)\]
\[= (n + 2)H_n + 1 + \frac{1}{n+1} - n - 1\]
\[= (n + 2)H_n - n + \frac{1}{n+1}.\]

That these two expressions are equal was precisely what we had to prove.

32. The statement is true for the base case, \(n = 0\), since \(3 \mid 0\). Suppose that \(3 \mid (k^3 + 2k)\). We must show that \(3 \mid ((k + 1)^3 + 2(k + 1))\). If we expand the expression in question, we obtain \(k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)\). By the inductive hypothesis, \(3\) divides \(k^3 + 2k\), and certainly \(3\) divides \(3(k^2 + k + 1)\), so \(3\) divides their sum, and we are done.

34. The statement is true for the base case, \(n = 0\), since \(6 \mid 0\). Suppose that \(6 \mid (n^3 - n)\). We must show that \(6 \mid ((n + 1)^3 - (n + 1))\). If we expand the expression in question, we obtain \(n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3n(n + 1)\). By the inductive hypothesis, \(6\) divides the first term, \(n^3 - n\). Furthermore clearly \(3\) divides the second term, and the second term is also even, since one of \(n\) and \(n + 1\) is even; therefore \(6\) divides the second term as well. This tells us that \(6\) divides the given expression, as desired. (Note that here we have, as promised, used \(n\) as the dummy variable in the inductive step, rather than \(k\).

36. It is not easy to stumble upon the trick needed in the inductive step in this exercise, so do not feel bad if you did not find it. The form is straightforward. For the basis step \((n = 1)\), we simply observe that \(4^{1+1} + 5^{2^1 - 1} = 16 + 5 = 21\), which is divisible by \(21\). Then we assume the inductive hypothesis, that \(4^{n+1} + 5^{2n-1}\) is divisible by \(21\), and let us look at the expression when \(n + 1\) is plugged in for \(n\). We want somehow to manipulate it so that the expression for \(n\) appears. We have
\[4^{(n+1)+1} + 5^{2(n+1)-1} = 4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1}\]
\[= 4 \cdot 4^{n+1} + (4 + 21) \cdot 5^{2n-1}\]
\[= 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1}.\]

Looking at the last line, we see that the expression in parentheses is divisible by \(21\) by the inductive hypothesis, and obviously the second term is divisible by \(21\), so the entire quantity is divisible by \(21\), as desired.

38. The basis step is trivial, as usual: \(A_j \subseteq B_j\) implies that \(\bigcup_{j=1}^1 A_j \subseteq \bigcup_{j=1}^1 B_j\) because the union of one set is itself. Assume the inductive hypothesis that if \(A_j \subseteq B_j\) for \(j = 1, 2, \ldots, k\), then \(\bigcup_{j=1}^k A_j \subseteq \bigcup_{j=1}^k B_j\). We want to show that if \(A_j \subseteq B_j\) for \(j = 1, 2, \ldots, k + 1\), then \(\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j\). To show that one set is a subset of another we show that an arbitrary element of the first set must be an element of the second set. So let \(x \in \bigcup_{j=1}^{k+1} A_j = \left(\bigcup_{j=1}^k A_j\right) \cup A_{k+1}\). Either \(x \in \bigcup_{j=1}^k A_j\) or \(x \in A_{k+1}\). In the first case we know by the inductive hypothesis that \(x \in \bigcup_{j=1}^k B_j\); in the second case, we know from the given fact that \(A_{k+1} \subseteq B_{k+1}\) that \(x \in B_{k+1}\). Therefore in either case \(x \in \left(\bigcup_{j=1}^k B_j\right) \cup B_{k+1} = \bigcup_{j=1}^{k+1} B_j\).

This is really easier to do directly than by using the principle of mathematical induction. For a noninductive proof, suppose that \(x \in \bigcup_{j=1}^n A_j\). Then \(x \in A_j\) for some \(j\) between 1 and \(n\), inclusive. Since \(A_j \subseteq B_j\), we know that \(x \in B_j\). Therefore by definition, \(x \in \bigcup_{j=1}^n B_j\).

40. If \(n = 1\) there is nothing to prove, and the \(n = 2\) case is the distributive law (see Table 1 in Section 2.2). Those take care of the basis step. For the inductive step, assume that
\[(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).\]