Announcements

• Quiz today!

• Homework Wednesday!

• Read Sections 6.1 (Basics of counting) and 6.2 (Pigeonhole principal) by Thursday
Two sets A and B have the same cardinality if and only if there exists a bijection (one-to-one correspondence) between them, $A \sim B$.

A set is “countable” if it is either finite or countably infinite.

An infinite set is “countably infinite” if it can be put into “one-to-one correspondence (bijection)” with the set of natural numbers.
Infinite Cardinality

- If there exists a function $f$ from $A$ to $B$ that is injective (i.e. one-to-one) we say that $|A| \leq |B|$

- If there exists a function $f$ from $A$ to $B$ that is surjective (i.e. onto) we say that $|A| \geq |B|$

Why?
Infinite Cardinality

Are there more evens than odds?

\(\{0, 2, 4, 6, 8, \ldots\} \sim \{1, 3, 5, 7, 9, \ldots\}, \ f(x) = x-1\)

Are there more natural numbers than evens?

\(\mathbb{N} \sim \{0, 2, 4, 6, 8, \ldots\}, \ f(x) = 2x\)

Are there more evens than multiples of 3?

\(\{0, 2, 4, 6, 8, \ldots\} \sim \{0, 3, 6, 9, 12, \ldots\}, \ f(x) = 3x/2\)
Infinite Cardinality

How many rational numbers are there?

\[ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]
\[ \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \ldots \]
\[ \frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}, \ldots \]

\ldots

\[ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \ldots \]
Infinite Cardinality

How many real numbers are in interval [0, 1]?

More irrational between 0 and 1 then all rational everywhere

"Countably many! There’s the list!"

"Are you sure they’re all there?"

Counterexample: 0.5 3 6 ... So we say the reals are “uncountable.”
Counting – Product Rule

– Suppose a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task and $n_2$ ways to do the second task, then there are $n_1 \times n_2$ ways to do the procedure.

$$|A \times B| = |A| \times |B|$$

If $A$ and $B$ are finite sets, the number of elements in the Cartesian product of the sets is product of the # elements in each set.
Counting – Sum Rule

If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, where none of the $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the task.

If $A$ and $B$ are disjoint sets then

$$|A \cup B| = |A| + |B|$$
Counting

Suppose you have 4 shirts, 3 pairs of pants, and 2 pairs of shoes. How many different outfits do you have?

Product Rule
Product Rule

How many functions are there from set A to set B?

To define each function we have to make 3 choices, one for each element of A.

4  4  4

How many ways can each choice be made?

64
Product Rule

How many one-to-one functions are there from set A to set B?

To define each function we have to make 3 choices, one for each element of A.

4  3  2

How many ways can each choice be made?

24
Counting

• Suppose a student can choose a computer project from one of three lists
  – List A – 23 projects
  – List B – 15 projects
  – List C – 19 projects

• No project is on more than one list

• How many possible projects are there to choose from?
  – Just count them up
  – \(23 + 15 + 19 = 57\)
Decision Tree (Tree Diagrams)

Suppose you have 4 shirts, 3 pairs of pants, and 2 pairs of shoes. How many different outfits do you have?
How many different best of 5 game series were possible between the Cubs and the Dodgers?

A DT is a good model for a sequence of events. It assists in counting, and can help you see special structure in the problem.
Inclusion Exclusion Principle

If a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, but some of the \( n_1 \) ways are the same as some of the \( n_2 \) ways.

Are there \( n_1 + n_2 \) ways to do the task now?

- **If sets A and B are NOT disjoint**
  \[
  |A \cup B| = |A| + |B| - |A \cap B|
  \]
Example

Count the number of bit strings of length 4 which begin with a 1 or end with 00

How many bit strings of length 4 begin with a 1?
• \(2^3 = 8\)

How many bit strings end in 00?
• \(2^2 = 4\)

So the total is \(8 + 4 = 12\) ?

How many bit strings end in 00 and start with 1 ?
• 2

The real total is \(8 + 4 - 2 = 10\)
Pigeonhole Principle

If \( n \) pigeons fly into \( k \) pigeonholes and \( k < n \), then some pigeonhole contains at least two pigeons.
Pigeonhole Principle

We can use this simple little fact to prove amazingly complex things.

If n pigeons fly into k pigeonholes and k < n, then some pigeonhole contains at least two pigeons.
Pigeonhole Principle

Let S contain any 6 positive integers. Then, there is a pair of numbers in S whose difference is divisible by 5.

Let S = \{a_1, a_2, a_3, a_4, a_5, a_6\}. Each of these has a remainder when divided by 5. What can these remainders be?

6 numbers, 5 possible remainders...what do we know?

Consider that pair, a_i and a_j, and their remainder r.

\[ a_i = 5m + r, \quad \text{and} \quad a_j = 5n + r. \]

Their difference: \[ a_i - a_j = (5m + r) - (5n + r) = 5m - 5n = 5(m-n), \] which is divisible by 5.
Pigeonhole Principle

Six people go to a party. Either there is a group of 3 who all know each other, or there is a group of 3 who are all strangers.

Consider one person.
She either knows or doesn’t know each other person.

But there are 5 other people! So, she knows, or doesn’t know, at least 3 others.

Let’s say she knows 3 others.
If any of those 3 know each other, we have a blue triangle, which means 3 people know each other. If not, then those 3 people must be strangers.

But then we’ve proven our conjecture for this case.

The case where she doesn’t know 3 others is similar.
Extra Examples
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In questions below suppose that a “word” is any string of seven letters of the alphabet, with repeated letters allowed.

1) How many words are there?

2) How many words end with the letter T?

3) How many words begin with R and end with T?

4) How many words begin with A or B?

5) How many words begin with A or end with B?

6) How many words begin with A or B and end with A or B?
Extra Examples Solutions

In questions below suppose that a “word” is any string of seven letters of the alphabet, with repeated letters allowed.

1) How many words are there? $26^7$

2) How many words end with the letter T? $26^6$

3) How many words begin with R and end with T? $26^5$

4) How many words begin with A or B? $2 * 26^6$

5) How many words begin with A or end with B?
   $26^6 + 26^6 - 26^5$

6) How many words begin with A or B and end with A or B?
   $4 * 26^5$