Announcements

- Quiz!
- HW!
- Exam will be graded this week
Probability (Section 7.1)

We roll a single die, what are the possible outcomes?

\{1,2,3,4,5,6\}

The set of possible outcomes is called the sample space.

We roll a pair of dice, what is the sample space?

\{(1,1),(1,2),(1,3),\ldots,(6,6)\}

Often convenient to choose a sample space of equally likely events.
Probability

Define a *probability measure* on a set $S$ to be a real-valued function, $Pr$, with domain $2^S$ so that:

1. For any subset $A$ in $2^S$, $0 \leq Pr(A) \leq 1$.
2. $Pr(\emptyset) = 0$, $Pr(S) = 1$.
3. If subsets $A$ and $B$ are disjoint, then $Pr(A \cup B) = Pr(A) + Pr(B)$.

$Pr(A)$ is “the probability of event $A$.”

A sample space, together with a probability measure, is called a *probability space*.

$S = \{1,2,3,4,5,6\}$

For $A \subseteq S$, $Pr(A) = |A|/|S|$

Ex. “Prob of an odd #”

$A = \{1,3,5\}$, $Pr(A) = 3/6$
Probability

Some things you already know:
If $A$ is a subset of $S$, let $\sim A$ be the complement of $A$ wrt $S$.

Then $\Pr(\sim A) = 1 - \Pr(A)$

If $A$ and $B$ are subsets of $S$, then

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Inclusion-Exclusion
What is the probability that a 5 card poker hand contains a royal flush?

\[ S = \text{all 5 card poker hands.} \]
\[ A = \text{all royal flushes} \]
\[ Pr(A) = \frac{|A|}{|S|} \]

\[ Pr(A) = \frac{4}{C(52,5)} \]
Probability

Which is more likely:

a) Rolling an 8 when 2 dice are rolled?
b) Rolling an 8 when 3 dice are rolled?
c) No clue.
What is the probability of a total of 8 when 2 dice are rolled?

What is the size of the sample space? 36

How many rolls satisfy our condition of interest? 5

So the probability is \( \frac{5}{36} \approx 0.139 \)
What is the probability of a total of 8 when 3 dice are rolled?

What is the size of the sample space?

How many rolls satisfy our condition of interest?

So the probability is \( \frac{21}{216} \approx 0.097 \)
Let $E$ and $F$ be events with $\Pr(F) > 0$. The conditional probability of $E$ given $F$, denoted by $\Pr(E | F)$ is defined to be:

$$\Pr(E | F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$
Conditional Probability

\[ Pr(E \mid F) = \frac{Pr(E \cap F)}{Pr(F)}. \]

A bit string of length 4 is generated at random so that each of the bit strings is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

How many different strings could exist?

\[ Pr(F) = \frac{1}{2} \]

\[ Pr(E \cap F)? \]

0000 0001 0010 0011 0100

\[ Pr(E \cap F) = \frac{5}{16} \]

\[ Pr(E \mid F) = \frac{5}{8} \]
Independence

The events $E$ and $F$ are independent if and only if $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$.

Let $E$ be the event that a family of $n$ children has children of both sexes.
Let $F$ be the event that a family of $n$ children has at most one boy.

Are $E$ and $F$ independent if $n = 2$?

No

$E = \{BG, GB\}$ $\quad F = \{GG, GB, BG\}$

$\Pr(E) = \frac{1}{2} \quad \Pr(F) = \frac{3}{4}$

$\Pr(E \cap F) = \frac{1}{2} \quad \Pr(E) \times \Pr(F) = \frac{3}{8}$
Independence

The events E and F are *independent* if and only if \( \Pr(E \cap F) = \Pr(E) \times \Pr(F) \).

Let E be the event that a family of n children has children of both sexes.

Lef F be the event that a family of n children has at most one boy.

Are E and F independent if

- \( n = 3 \)? Yes

E = \{GGB, GBG, GBB, BGG, BGB, BBG\}

F = \{GGG, GGB, GBG, BGG\}

\( \Pr(E) = 3/4 \) \hspace{1cm} \( \Pr(F) = 1/2 \)

\( \Pr(E \cap F) = 3/8 \) \hspace{1cm} \( \Pr(E) \times \Pr(F) = 3/8 \)
Independence

The events E and F are **independent** if and only if $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$.

Let E be the event that a family of n children has children of both sexes.

Let F be the event that a family of n children has at most one boy.

Are E and F independent if

$n = 4$?

No

16 ways to have 4 children \((2^4)\)

\[
\begin{align*}
\Pr(E) &= 14/16 = 7/8 \\
\Pr(F) &= 5/16 \\
\Pr(E \cap F) &= 4/16 = 1/4 \\
\Pr(E) \times \Pr(F) &= 35/128
\end{align*}
\]
Independence

The events E and F are *independent* if and only if $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$.

Let $E$ be the event that a family of $n$ children has children of both sexes. Let $F$ be the event that a family of $n$ children has at most one boy.

Are $E$ and $F$ independent if

$n = 5$? No
Independence

The events E and F are *independent* if and only if $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$.

Let E be the event that a family of n children has children of both sexes.
Let F be the event that a family of n children has at most one boy.

Are E and F independent if

- $n = 2$? No
- $n = 3$? Yes
- $n = 4$? No
- $n = 5$? No
Birthdays

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than 1/2?

Let $p_n$ be the probability that no people share a birthday among $n$ people in a room.

Then $1 - p_n$ is the probability that 2 or more share a birthday.

We want the smallest $n$ so that $1 - p_n > 1/2$.
Assume 366 days in a year

$$1 - p_n = 1 - \frac{365}{366} \frac{364}{366} \frac{363}{366} \ldots \frac{367 - n}{366}$$

$n = 23$

$1 - p_n \approx 0.506$
Bernoulli Trials

- A coin is tossed 8 times. What is the probability of exactly 3 heads in the 8 tosses?

- THHTTHTT is a tossing sequence...

- How many ways of choosing 3 positions for the heads?

- What is the probability of a particular sequence?

- In general: The probability of exactly $k$ successes in $n$ independent Bernoulli trials with probability of success $p$, is

$$C(n,k)p^k(1-p)^{n-k}$$
Bernoulli Trials

A game of Jewel Quest is played 5 times. You clear the board 70% of the time. What is the probability that you win a majority of the 5 games?

Sanity check: What is the probability that the result is WWLLW?

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is

$$C(n,k)p^k(1-p)^{n-k}$$

$$C(5,3)0.7^30.3^2 + C(5,4)0.7^40.3^1 + C(5,5)0.7^50.3^0$$
Random Variables

For a given sample space $S$, a *random variable* is any real valued function on $S$.

Suppose our experiment is a roll of 2 dice. $S$ is set of pairs.

- $X = \text{sum of two dice.}$  
  \[X((2,3)) = 5\]
- $Y = \text{difference between two dice.}$  
  \[Y((2,3)) = 1\]
- $Z = \text{max of two dice.}$  
  \[Z((2,3)) = 3\]
Random Variables

Example:
Suppose we are playing a game with cards labeled 1 to 20, and we draw 3 cards. We bet that the maximum card has value 17 or greater. What’s the probability we win the bet?

Let r.v. $X$ denote the maximum card value. The possible values for $X$ are 3, 4, 5, ..., 20.

<table>
<thead>
<tr>
<th>$i$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>...</th>
<th>20</th>
</tr>
</thead>
</table>

Filling in this box would be a pain. We look for a general formula.
Random Variables

X is value of the highest card among the 3 selected. 20 cards are labeled 1 through 20.

We want \( \Pr(X = i) \), \( i = 3, \ldots, 20 \).

Denominator first: How many ways are there to select the 3 cards?

\[ C(20,3) \]

How many choices are there that result in a max card whose value is \( i \)?

\[ C(i-1,2) \]

\( \Pr(X = i) = \frac{C(i-1,2)}{C(20,3)} \)

These are the table values.

We win the bet if the max card, \( X \) is 17 or greater. What’s the probability we win?

\[ \Pr(X = 17) + \Pr(X = 18) + \Pr(X = 19) + \Pr(X = 20) \approx 0.51 \]
Practice Problem

A class has 20 women and 13 men. A committee of five is chosen at random. Find
(a) \( p \) (the committee consists of all women).
(b) \( p \) (the committee consists of all men)
(c) \( p \) (the committee consists of all of the same sex)
1) A class has 20 women and 13 men. A committee of five is chosen at random.
(a) $\frac{C(20,5)}{C(33,5)}$
(b) $\frac{C(13,5)}{C(33,5)}$
(c) $\frac{C(20,5) + C(13,5)}{C(33,5)}$
Quiz