Announcements

- Quiz today
- HW on Wednesday
- Class the week of Thanksgiving
Expected Value

Definition: The expected value (or expectation or mean) of the random variable $X$ is

$$\sum_v p(X = v) \cdot v$$

Example-Expected Value of a Die: Let $X$ be the number that comes up when a fair die is rolled. What is the expected value of $X$?

Solution: The random variable $X$ takes the values 1, 2, 3, 4, 5, or 6. Each has probability $1/6$. It follows that

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots + \frac{1}{6} \cdot 6 = \frac{21}{6} = \frac{7}{2}.$$
Theorem: The expected number of successes when $n$ mutually independent Bernoulli trials are performed, where, the probability of success on each trial is $p$ is equal to $np$.

Proof: Let $X$ be the random variable equal to the number of success in $n$ trials.

We know that $p(X = k) = C(n,k)p^kq^{n-k}$ by the definition of Bernoulli trials.

Thus, 

$$E(X) = \sum_{k=1}^{n} kp(X = k)$$
Expected Value

\[ E(X) = \sum_{k=1}^{n} kp(X = k) \]

\[ = \sum_{k=1}^{n} kC(n, k)p^k q^{n-k} \quad \text{By def. of Bernoulli trials} \]

\[ = \sum_{k=1}^{n} nC(n-1, k-1)p^k q^{n-k} \quad \text{Why? Exercise} \]

\[ = np \sum_{k=1}^{n-1} C(n-1, k-1)p^{k-1} q^{n-k} \]

\[ = np \sum_{j=0}^{n-1} C(n-1, j)p^j q^{n-1-j} \quad \text{Shift index} \]

\[ = np(p + q)^{n-1} \]

\[ = np. \quad \text{Is there a better way?} \]
Linearity of Expectation

**Theorem 3:** If $X_i$, $i = 1, 2, \ldots, n$ with $n$ a positive integer, are random variables on $S$, and if $a$ and $b$ are real numbers, then

(i) $E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$

(ii) $E(aX + b) = aE(X) + b$. 
Linearity of Expectation

**Theorem**: The expected number of successes when \( n \) mutually independent Bernoulli trials are performed, where, the probability of success on each trial is \( p \) is equal to \( np \).

**Different Proof**: Let \( X_i \) be 1 if the \( i \)th trial is a success and 0 otherwise. Let \( X \) be the total number of successes. Note that \( X = \Sigma_i X_i \)

Note that \( E(X_i) = p \)

By the linearity of expectation:

\[
E(X) = \Sigma_i E(X_i) = \Sigma_i p = np
\]
Linearity of Expectation

Expected Value in the Hatcheck Problem: A new employee started a job checking hats, but forgot to put the claim check numbers on the hats. So, the $n$ customers just receive a random hat from those remaining. What is the expected number of hats returned correctly?
Linearity of Expectation

Solution: Let $X$ be the random variable that equals the number of people who receive the correct hat. Note that $X = X_1 + X_2 + \cdots + X_n$, where $X_i = 1$ if the $i$th person receives the hat and $X_i = 0$ otherwise.

- Because it is equally likely that the checker returns any of the hats to the $i$th person, it follows that the probability that the $i$th person receives the correct hat is $1/n$. Consequently, for all $i$
  
  $$E(X_i) = 1 \cdot p(X_i = 1) + 0 \cdot p(X_i = 0) = 1 \cdot 1/n + 0 = 1/n.$$ 

- By the linearity of expectations, it follows that:
  
  $$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = n \cdot 1/n = 1.$$ 

Consequently, the average number of people who receive the correct hat is exactly 1. (Surprisingly, this answer remains the same no matter how many people have checked their hats!)
Linearity of Expectation

**Expected Number of Inversions in a Permutation:** The ordered pair \((i, j)\) is an *inversion* in a permutation of the first \(n\) positive integers if \(i < j\), but \(j\) precedes \(i\) in the permutation.

*Example:* There are six inversions in the permutation of 3, 5, 1, 4, 2 \((1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5)\).

Find the average (expected) number of inversions in a random permutation of the first \(n\) integers.
Linearity of Expectation

Solution: Let $I_{i,j}$ be the random variable on the set of all permutations of the first $n$ positive integers with $I_{i,j} = 1$ if $(i,j)$ is an inversion of the permutation and $I_{i,j} = 0$ otherwise. If $X$ is the random variable equal to the number of inversions in the permutation, then

$$X = \sum_{1 \leq i < j \leq n} I_{i,j}.$$

- Since it is equally likely for $i$ to precede $j$ in a randomly chosen permutation as it is for $j$ to precede $i$, we have: $E(I_{i,j}) = 1 \cdot p(I_{i,j} = 1) + 0 \cdot p(I_{i,j} = 0) = 1 \cdot 1/2 + 0 = 1/2$, for all $(i,j)$.

- Because there are $C(n,2)$ pairs $i$ and $j$ with $1 \leq i < j \leq n$, by the linearity of expectation, we have:

$$E(X) = \sum_{1 \leq i < j \leq n} E(I_{i,j}) = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n - 1)}{2} \cdot \frac{1}{2}.$$

Consequently, it follows that there is an average of $n(n - 1)/4$ inversions in a random permutation of the first $n$ positive integers.
Independent Variables

Recall that the random variables $X$ and $Y$ on a sample space $S$ are independent if

$$p(X = r_1 \text{ and } Y = r_2) = p(X = r_1) \cdot p(Y = r_2).$$

**Theorem 5:** If $X$ and $Y$ are independent variables on a sample space $S$, then $E(XY) = E(X)E(Y)$.

Proof: Exercise
Markov’s Inequality

We often want to know how close a random variable is to its expectation. Is it far or close?

A generic answer, for a non-negative, random variable is Markov’s inequality

Theorem: For any non-negative random variable $X$

$$P(X \geq a) \leq \frac{E(X)}{a}$$
Markov’s Inequality

Proof:

\[ E(X) = \sum_{x} p(X = x) \cdot x \]

\[ = \sum_{x < a} p(X = x) \cdot x + \sum_{x \geq a} p(X = x) \cdot x \]

\[ \geq \sum_{x \geq a} p(X = x) \cdot a \]

Divide by ‘a’ to get Markov’s inequality

X is non-negative
Markov’s Inequality Example

Question: A biased coin, which lands heads with probability 1/10 each time it is flipped, is flipped 200 times consecutively. Give an upper bound on the probability that it lands heads at least 120 times.
Markov’s Inequality Example

Question: A biased coin, which lands heads with probability 1/10 each time it is flipped, is flipped 200 times consecutively. Give an upper bound on the probability that it lands heads at least 120 times.

Solution: The number of heads is a binomially distributed random variable, X, with parameter \( p = 1/10 \) and \( n=200 \).

The expected number of heads is \( E(X)= np = 200*(1/10) = 20 \)

By Markov’s inequality, the probability of at least 120 heads is

\[ P(X \geq 120) \leq \frac{E(X)}{120} = \frac{20}{120} = \frac{1}{6} \]

Note that this is just an upper bound. Markov can be very loose.
Variance

**Deviation:** The *deviation* of $X$ at $s \in S$ is $X(s) - E(X)$, the difference between the value of $X$ and the mean of $X$.

**Definition 4:** Let $X$ be a random variable on the sample space $S$. The *variance* of $X$, denoted by $V(X)$ is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

That is $V(X)$ is the weighted average of the square of the deviation of $X$. The standard deviation of $X$, denoted by $\sigma(X)$ is defined to be the square root of the variance.

**Theorem 6:** If $X$ is a random variable on a sample space $S$, then $V(X) = E(X^2) - E(X)^2$.

**Corollary 1:** If $X$ is a random variable on a sample space $S$ and $E(X) = \mu$, then $V(X) = E((X - \mu)^2)$. [See book for a proof]


**Variance**

**Example:** What is the variance of the random variable \( X \), where \( X(t) = 1 \) if a Bernoulli trial is a success and \( X(t) = 0 \) if it is a failure, where \( p \) is the probability of success and \( q \) is the probability of failure?

**Solution:** Because \( X \) takes only the values 0 and 1, it follows that \( X^2(t) = X(t) \). Hence,

\[
V(X) = E(X^2) - E(X)^2 = p - p^2 = p(1 - p) = pq.
\]

**Variance of the Value of a Die:** What is the variance of a random variable \( X \), where \( X \) is the number that comes up when a fair die is rolled?

**Solution:** We have \( V(X) = E(X^2) - E(X)^2 \). In an earlier example, we saw that \( E(X) = 7/2 \). Note that

\[
E(X^2) = 1/6(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = 91/6.
\]

We conclude that \( V(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} \).
Variance

**Bienaymé's Formula**: If $X$ and $Y$ are two independent random variables on a sample space $S$, then $V(X + Y) = V(X) + V(Y)$. Furthermore, if $X_i$, $i = 1, 2, \ldots, n$, with $n$ a positive integer, are pairwise independent random variables on $S$, then

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n).$$

See text for proof
Variance

Example: Find the variance of the number of successes when \( n \) independent Bernoulli trials are performed, where on each trial, \( p \) is the probability of success and \( q \) is the probability of failure.

Solution: Let \( X_i \) be the random variable with \( X_i = 1 \) if trial \( t_i \) is a success and \( X_i = 0 \) if it is a failure. Let \( X = X_2 + X_3 + \ldots + X_n \). Then \( X \) counts the number of successes in the \( n \) trials.

- By Bienaymé ‘s Formula, it follows that \( V(X) = V(X_1) + V(X_2) + \ldots + V(X_n) \).
- By the previous example, \( V(X_i) = pq \) for \( i = 1, 2, \ldots, n \).

Hence, \( V(X) = npq \).
Chebyshev’s Inequality

**Chebyshev’s Inequality:** Let $X$ be a random variable on a sample space $S$ with probability function $p$. If $r$ is a positive real number, then

$$p(|X(s) - E(X)| \geq r) \leq \frac{V(X)}{r^2}.$$
Example: Suppose that $X$ is a random variable that counts the number of tails when a fair coin is tossed $n$ times. Note that $X$ is the number of successes when $n$ independent Bernoulli trials, each with probability of success $\frac{1}{2}$ are done. Hence, $E(X) = n/2$ and $V(X) = n/4$.

By Chebyshev’s inequality with $r = \sqrt{n}$,

$$p(|X(s) - n/2| \geq \sqrt{n}) \leq \left(\frac{n/4}{\sqrt{n}}\right)^2 = \frac{1}{4}.$$  

This means that the probability that the number of tails that come up on $n$ tosses deviates from the mean, $n/2$, by more than $\sqrt{n}$ is no larger than $\frac{1}{4}$.  

Chebyshev’s Inequality