Announcements

• Homework 1 is posted online and due next Wednesday
• Problems due next Wednesday
  – HW and Quiz problems are also posted online

• Quiz Week from Monday
  – I will choose one problem from this set
Reading

• Read Section 1.4 (Predicates and Quantifiers) and read 1.5 (Nested Quantifiers)
Propositional Logic (Section 1)

- Proofs involve stepping through a mathematical argument
- Propositional Logic provides such steps
- Today we will discuss the process of moving from one proposition to the next to form a mathematical argument
Logical Equivalence

- Challenge: Try to find a proposition that is equivalent to $p \rightarrow q$, but that uses only the connectives $\neg$, $\land$, and $\lor$

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<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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- $p \rightarrow q$ is logically equivalent to $\neg p \lor q$
- or $p \rightarrow q \equiv \neg p \lor q$
Are these equivalent?

• **Contrapositives**: \( p \rightarrow q \) and \( \neg q \rightarrow \neg p \)
  
  Ex. “If it is noon, then I am hungry.”
  “If I am not hungry, then it is not noon.”

• **Converses**: \( p \rightarrow q \) and \( q \rightarrow p \)
  
  Ex. “If it is noon, then I am hungry.”
  “If I am hungry, then it is noon.”

• **Inverses**: \( p \rightarrow q \) and \( \neg p \rightarrow \neg q \)
  
  Ex. “If it is noon, then I am hungry.”
  “If it is not noon, then I am not hungry.”
Are these equivalent?

- **Contrapositives**: $p \rightarrow q \equiv \neg q \rightarrow \neg p$ ?
  
  Ex. “If it is noon, then I am hungry.”
  “If I am not hungry, then it is not noon.”

  YES

- **Converses**: $p \rightarrow q \equiv q \rightarrow p$ ?
  
  Ex. “If it is noon, then I am hungry.”
  “If I am hungry, then it is noon.”

  NO

- **Inverses**: $p \rightarrow q \equiv \neg p \rightarrow \neg q$ ?
  
  Ex. “If it is noon, then I am hungry.”
  “If it is not noon, then I am not hungry.”

  NO
Definitions

A **tautology** is a proposition that’s always TRUE.

A **contradiction** is a proposition that’s always FALSE.

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<th>¬p</th>
<th>p ∨ ¬p</th>
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Logical Equivalences

- **Identity**
  
  $p \land T \equiv p$

  $p \lor F \equiv p$

- **Domination**
  
  $p \lor T \equiv T$

  $p \land F \equiv F$

- **Idempotent**
  
  $p \lor p \equiv p$

  $p \land p \equiv p$
Logical Equivalences Continued

• **Excluded Middle** \( p \lor \neg p \equiv T \)

• **Uniqueness** \( p \land \neg p \equiv F \)

• **Double negation** \( \neg(\neg p) \equiv p \)
Logical Equivalences Continued

- **Commutativity**
  
  \[ p \lor q = q \lor p \]
  
  \[ p \land q = q \land p \]

- **Associativity**
  
  \[(p \lor q) \lor r = p \lor (q \lor r)\]
  
  \[(p \land q) \land r = p \land (q \land r)\]

- **Distributivity**
  
  \[ p \lor (q \land r) = (p \lor q) \land (p \lor r) \]
  
  \[ p \land (q \lor r) = (p \land q) \lor (p \land r) \]
### Proof of Distributivity

$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

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<th>$p \lor (q \land r)$</th>
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<th>$p \lor r$</th>
<th>$(p \lor q) \land (p \lor r)$</th>
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DeMorgan’s

- **De Morgan’s I** \( \neg (p \lor q) \equiv \neg p \land \neg q \)

- **De Morgan’s II** \( \neg (p \land q) \equiv \neg p \lor \neg q \)
Example of De Morgan’s

\[ \neg (p \lor q) \equiv \neg p \land \neg q \]

\[ \neg (p \land q) \equiv \neg p \lor \neg q \]
De Morgan’s Continued

- **De Morgan’s II**

\[ \neg(p \land q) \equiv \neg p \lor \neg q \]

\[\neg(p \land q) \equiv \neg(\neg\neg p \land \neg\neg q) \quad \text{Double negation} \]

\[\equiv (\neg p \lor \neg q) \quad \text{De Morgan’s I} \]

\[\equiv (\neg p \lor \neg q) \quad \text{Double negation} \]
Proof equivalence

if NOT (blue AND NOT red) OR red then...

\[ \neg(p \land \neg q) \lor q \equiv \neg p \lor q \]

De Morgan’s II

\[ \equiv (\neg p \lor \neg q) \lor q \]

Double negation

\[ \equiv \neg p \lor (q \lor q) \]

Associativity

\[ \equiv \neg p \lor q \]

Idempotent
Another Example

Show that \([p \land (p \rightarrow q)] \rightarrow q\) is a tautology.

We use \(\equiv\) to show that \([p \land (p \rightarrow q)] \rightarrow q \equiv T\)

\[
[p \land (p \rightarrow q)] \rightarrow q
\]

\[\equiv [p \land (\neg p \lor q)] \rightarrow q\] substitution for \(\rightarrow\)

\[\equiv [(p \land \neg p) \lor (p \land q)] \rightarrow q\] distributive

\[\equiv [F \lor (p \land q)] \rightarrow q\] uniqueness

\[\equiv (p \land q) \rightarrow q\] identity

\[\equiv \neg (p \land q) \lor q\] substitution for \(\rightarrow\)

\[\equiv (\neg p \lor \neg q) \lor q\] De Morgan’s II

\[\equiv \neg p \lor q\] associative

\[\equiv \neg p \lor T\] excluded middle

\[\equiv \neg p \lor (\neg q \lor q)\] domination

\[\equiv \neg p \lor T\]

\[\equiv T\]
Another Example

- Consider the newspaper headline: “Legislature Fails to Override Governor’s Veto of Bill to Cancel Sales Tax Reform”

Did the legislature vote in favor of or against the sales tax reform?
Another Example

• Consider the newspaper headline:
  “Legislature Fails to Override Governor’s Veto of Bill to Cancel Sales Tax Reform”

Did the legislature vote in favor of or against the sales tax reform?

A) The legislature DID vote in favor
B) The legislature DID NOT vote in favor

Let’s take a vote
Another Example

- Consider the newspaper headline: “Legislature Fails to Override Governor’ s Veto of Bill to Cancel Sales Tax Reform”

Did the legislature vote in favor of or against the sales tax reform?

Let s stand for “sales tax reform”

Unravel one at a time.

- The bill to cancel sales tax reform is ¬s
- The governors veto of the bill is ¬ ¬s
- Overriding this means ¬ ¬ ¬s

Therefore the legislature does not support sales tax reform. He voted in favor of the bill (to cancel it).
Group Exercise I

• Break up into groups of 2 or 3 people

• Answer the following question:

• Prove De Morgan’s I Law by manipulating symbols (not a truth table)

\[ \neg(p \lor q) \equiv \neg p \land \neg q \]

• Hint- Similar to the proof of De Morgan’s II Law

• Reminder
  – DeMorgan’s II

\[ \neg(p \land q) \equiv \neg p \lor \neg q \]
Predicate Logic

Proposition, YES or NO?

$3 + 2 = 5$  
Yes

$X + 2 = 5$  
No

$X + 2 = 5$ for all choices of $X$ in $\{1, 2, 3\}$  
Yes

$X + 2 = 5$ for some choice of $X$ in $\{1, 2, 3\}$  
Yes
Alicia eats pizza at least once a week.

Define:

\[ EP(x) = "x \text{ eats pizza at least once a week}" \]

Universe of Discourse - x is a student in cs240

A *predicate*, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Note that \( EP(x) \) is not a proposition, \( EP(Ariel) \) is.
Suppose $Q(x,y) = “x > y”$

**Proposition, YES or NO?**

<table>
<thead>
<tr>
<th>$Q(x,y)$</th>
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<tbody>
<tr>
<td>$Q(3,4)$</td>
<td>Yes</td>
</tr>
<tr>
<td>$Q(x,9)$</td>
<td>No</td>
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**Predicate, YES or NO?**

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Universal Quantifier

Another way of changing a predicate into a proposition.

Suppose $P(x)$ is a predicate on some universe of discourse.
Ex. $B(x) = \text{“x is carrying a backpack, ” x is set of cs240 students.}$

The universal quantifier of $P(x)$ is the proposition:
“$P(x)$ is true for all $x$ in the universe of discourse.”

We write it $\forall x \ P(x)$, and say “for all $x$, $P(x)$”

$\forall x \ P(x)$ is TRUE if $P(x)$ is true for every single $x$.
$\forall x \ P(x)$ is FALSE if there is an $x$ for which $P(x)$ is false.
Extensible Networking Platform

B(x) = “x is enrolled in CS240.”
L(x) = “x is signed up for at least one class.”
Y(x) = “x is wearing sneakers.”

Are either of these propositions true?

a) \( \forall x \ (B(x) \rightarrow Y(x)) \)

b) \( \forall x \ (Y(x) \lor L(x)) \)

What does this proposition mean?
\( \forall x \ (Y(x) \land B(x)) \)

Is it true?

A: only a is true
B: only b is true
C: both are true
D: neither is true

Universe of discourse is people in this room.
Existential Quantifier

Another way of changing a predicate into a proposition.

Suppose \( P(x) \) is a predicate on some universe of discourse.

Ex. \( C(x) = \) “\( x \) has a candy bar,” \( x \) is the set of cs240 students.

The existential quantifier of \( P(x) \) is the proposition:

“\( P(x) \) is true for some \( x \) in the universe of discourse.”

We write it \( \exists x \ P(x) \), and say “for some \( x \), \( P(x) \)”

\( \exists x \ P(x) \) is TRUE if there is an \( x \) for which \( P(x) \) is true.
\( \exists x \ P(x) \) is FALSE if \( P(x) \) is false for every single \( x \).
Existential Quantifier

B(x) = “x is enrolled in CS240.”  
L(x) = “x is signed up for at least one class.”

Are either of these propositions true?

a) \( \exists x \ B(x) \)

b) \( \exists x \ (B(x) \land L(x)) \)

What does this proposition mean?

\( \exists x \ (B(x) \rightarrow L(x)) \)

Why do I have to be careful with this proposition?

A: only a is true  
B: only b is true  
C: both are true  
D: neither is true

Universe of discourse is people in this room.
Predicate Examples

L(x) = “x is a lion.”
F(x) = “x is fierce.”
C(x) = “x drinks coffee.”

All lions are fierce.

Some lions don’t drink coffee.

Some fierce creatures don’t drink coffee.

Universe of discourse is all creatures.

∀x (L(x) → F(x))

∃x (L(x) ∧ ¬C(x))

∃x (F(x) ∧ ¬C(x))
B(x) = “x is a hummingbird.”
L(x) = “x is a large bird.”
H(x) = “x lives on honey.”
R(x) = “x is richly colored.”

All hummingbirds are richly colored.

No large birds live on honey.

Birds that do not live on honey are dully colored.
More Examples

S(x) = “x is a student in this class.”
C(x) = “x has visited Chicago.”

∀x (S(x) ∧ C(x))
All people are students in this class and all people have visited Chicago

∀x (S(x) → C(x))
All students in this class have visited Chicago

∃x (S(x) → C(x))
Some student in this class has visited Chicago

∃x (S(x) ∧ C(x))
Not all large birds live on honey.

\( \forall x \ P(x) \) means “\( P(x) \) is true for every \( x \).”

What about \( \neg \forall x \ P(x) \)?

Not [“\( P(x) \) is true for every \( x \).”]

“There is an \( x \) for which \( P(x) \) is not true.”

\( \exists x \, \neg P(x) \)

So, \( \neg \forall x \ P(x) \) is the same as \( \exists x \, \neg P(x) \).
Quantifier Negation

No large birds live on honey.

\( \exists x \, P(x) \) means “P(x) is true for some x.”

What about \( \neg \exists x \, P(x) \)?

Not [“P(x) is true for some x.”]

“P(x) is not true for all x.”

\( \forall x \, \neg P(x) \)

So, \( \neg \exists x \, P(x) \) is the same as \( \forall x \, \neg P(x) \).
So, $\neg \forall x P(x)$ is the same as $\exists x \neg P(x)$.

So, $\neg \exists x P(x)$ is the same as $\forall x \neg P(x)$.

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.