Announcements

- Homework 1 is due Today
- Quiz 1 is next Monday
- Read Section 1.6 (Rules of Inference), 1.7 (Introduction to Proofs) and 1.8 (Proof Methods and Strategy) by Thursday
Another way of changing a predicate into a proposition.

Suppose $P(x)$ is a predicate on some universe of discourse.
Ex. $B(x) = \text{“x is carrying a backpack,” x is set of cs240 students.}$

The universal quantifier of $P(x)$ is the proposition:
“$P(x)$ is true for all $x$ in the universe of discourse.”

We write it $\forall x P(x)$, and say “for all $x$, $P(x)$”

$\forall x P(x)$ is TRUE if $P(x)$ is true for every single $x$.
$\forall x P(x)$ is FALSE if there is an $x$ for which $P(x)$ is false.

$\forall x B(x)$?

What is this asking?

Is it true or false?
Existential Quantifier

Another way of changing a predicate into a proposition.

Suppose $P(x)$ is a predicate on some universe of discourse.

Ex. $C(x) = \text{“}x\text{ has a candy bar,”}$ $x$ is the set of cs240 students.

The existential quantifier of $P(x)$ is the proposition:

$P(x)$ is true for some $x$ in the universe of discourse.”

We write it $\exists x \ P(x)$, and say “for some $x$, $P(x)$”

$\exists x \ P(x)$ is TRUE if there is an $x$ for which $P(x)$ is true.

$\exists x \ P(x)$ is FALSE if $P(x)$ is false for every single $x$. 

$\exists C(x)$?

What is this asking?

Is it true or false?
Predicate Examples

L(x) = “x is a lion.”
F(x) = “x is fierce.”
C(x) = “x drinks coffee.”

All lions are fierce.

Some lions don’t drink coffee.

Some fierce creatures don’t drink coffee.
More Examples

$B(x) = \text{“x is a hummingbird.”} \quad \text{Universe of discourse is all birds.}$

$L(x) = \text{“x is a large bird.”}$

$H(x) = \text{“x lives on honey.”}$

$R(x) = \text{“x is richly colored.”}$

All hummingbirds are richly colored.

$\forall x \ (B(x) \rightarrow R(x))$

No large birds live on honey.

$\neg \exists x \ (L(x) \land H(x))$

Birds that do not live on honey are dully colored.

$\forall x \ (\neg H(x) \rightarrow \neg R(x))$
More Examples

$S(x) = “x \text{ is a student in this class."}$
$C(x) = “x \text{ has visited Chicago.}”$

Universe of discourse is all people.

$\forall x \ (S(x) \land C(x))$
All people are students in this class and all people have visited Chicago

$\forall x \ (S(x) \rightarrow C(x))$
All students in this class have visited Chicago

$\exists x \ (S(x) \rightarrow C(x))$

Some student in this class has visited Chicago

$\exists x \ (S(x) \land C(x))$

?
Quantifier Negation

Not all large birds live on honey.

∀x P(x) means “P(x) is true for every x.”

What about ¬∀x P(x) ?

Not [“P(x) is true for every x.”]

“There is an x for which P(x) is not true.”

∃x ¬P(x)

So, ¬∀x P(x) is the same as ∃x ¬P(x).
Quantifier Negation

No large birds live on honey.

\( \exists x \ P(x) \) means “\( P(x) \) is true for some \( x \).”

What about \( \neg \exists x \ P(x) \)?

Not [“\( P(x) \) is true for some \( x \).”]

“\( P(x) \) is not true for all \( x \).”

\( \forall x \ \neg P(x) \)

So, \( \neg \exists x \ P(x) \) is the same as \( \forall x \ \neg P(x) \).
Quantifier Negation

So, \( \neg \forall x \, P(x) \) is the same as \( \exists x \, \neg P(x) \).

So, \( \neg \exists x \, P(x) \) is the same as \( \forall x \, \neg P(x) \).

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.
Predicates – Multiple Quantifiers

\( \forall x \forall y \ P(x, y) \)  \hspace{1cm} P(x, y) \text{ true for all } x, \ y \text{ pairs.}

\( \exists x \exists y \ P(x, y) \)  \hspace{1cm} P(x, y) \text{ true for at least one } x, \ y \text{ pair.}

\( \forall x \exists y \ P(x, y) \)  \hspace{1cm} \text{For every value of } x \text{ we can find a (possibly different) } y \text{ so that } P(x, y) \text{ is true.}

\( \exists x \forall y \ P(x, y) \)  \hspace{1cm} \text{There is at least one } x \text{ for which } P(x, y) \text{ is true for every } y. \)
Predicates – multiple quantifiers

N(x,y) = “x is sitting by y”

∀x∀y N(x,y)  False
∃x∃y N(x,y)  True
∀x∃y N(x,y)  True
∃x∀y N(x,y)  False

Universe of discourse is all students in this room.

Let “sitting by” be defined as x is sitting within 5 feet of y.
A **theorem** is a statement that can be shown to be true.

A **proof** is the means of doing so.
Proofs - how do you know?

The following statements are true:

If I am Mila, then I am a great swimmer.
I am Mila.

What do we know to be true?

I am a great swimmer!

What rule of inference can we use to justify it?
Proofs - Definitions

• **Argument**
  – A sequence of propositions
  – All but the final proposition are called premises
    • Final proposition called conclusion
  – An argument is valid if the truth of all premises implies the conclusion is true

• **Argument form**
  – Sequence of compound propositions involving proposition variables

premise \[ p \]
premise \[ p \rightarrow q \]

conclusion \[ \therefore q \]
I am Mila.
If I am Mila, then I am a great swimmer.

\[ \therefore \text{I am a great swimmer!} \]

\( p \rightarrow q \)

Tautology:
\( (p \land (p \rightarrow q)) \rightarrow q \)

Inference Rule:
Modus Ponens

Modus Ponens is Latin for "the way that affirms by affirming"
I am not a great skater. 
If I am Erik, then I am a great skater.

∴ I am not Erik!

¬q
p → q

∴ ¬p

Tautology:
(¬q ∧ (p → q)) → ¬p

Inference Rule:
Modus Tollens

Modus Tollens is Latin for "the way that denies by denying"
I am not a great skater.

∴ I am not a great skater or I am a monkey.

\[ p \implies (p \lor q) \]

Tautology:

Inference Rule:
Addition or Weakening
I am not a great skater and you are sleepy.

\[ p \land q \rightarrow p \]

**Tautology:**

**Inference Rule:**

Simplification
Proofs - Disjunctive Syllogism

I am a great eater or I am a great skater.
I am not a great skater.

∴ I am a great eater!

\[ p \lor q \]
\[ \neg q \]
\[ \therefore p \]

Tautology:
\[ ((p \lor q) \land \neg q) \rightarrow p \]

Inference Rule:
Disjunctive Syllogism
If you are an athlete, you are always hungry. If you are always hungry, you have a snickers in your backpack.

∴ If you are an athlete, you have a snickers in your backpack.

\[ p \rightarrow q \]
\[ q \rightarrow r \]
\[ \therefore p \rightarrow r \]

Tautology:
\[ ((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r) \]

Inference Rule:
Hypothetical Syllogism
I am a great athlete
I am always hungry

\[ \therefore \text{ I am a great athlete and always hungry} \]

\[ p \land q \]

\[ \text{Tautology: } \left( (p) \land (q) \right) \rightarrow (p \land q) \]

\[ \text{Inference Rule: Conjunction} \]
Proofs - fallacies

Rules of inference, appropriately applied give *valid* arguments.

Mistakes in applying rules of inference are called *fallacies*. 
Valid Argument or Fallacy?

If I am Bonnie Blair, then I skate fast
I skate fast!

∴ I am Bonnie Blair

I’m Eric Heiden

If you don’t give me $10, I bite your ear.
I bite your ear!

∴ You didn’t give me $10.

I’m just mean.

Affirming the conclusion.

((p → q) ∧ q) → p
Not a tautology.
Valid Argument or Fallacy?

If it rains then it is cloudy.
It does not rain.

\[ \vdash \text{It is not cloudy} \]

If it is a car, then it has 4 wheels.
It is not a car.

\[ \vdash \text{It doesn’t have 4 wheels.} \]

Denying the hypothesis.

\[ ((p \rightarrow q) \land \neg p) \rightarrow \neg q \]
Not a tautology.

January!
Here’s what you know:

- Pat is a math major or a CS major.
- If Pat does not like discrete math, Pat is not a CS major.
- If Pat likes discrete math, Pat is smart.
- Pat is not a math major.

Can you conclude Pat is smart?

\[ \begin{align*}
M \lor C \land (\neg D \rightarrow \neg C) \land (D \rightarrow S) \land (\neg M) & \rightarrow S \\
\end{align*} \]
Proof Techniques - direct proofs

In general, to prove $p \rightarrow q$, assume $p$ and show that $q$ follows.

$((M \lor C) \land (\neg D \rightarrow \neg C) \land (D \rightarrow S) \land (\neg M)) \rightarrow S$?
Proof Techniques - direct proofs

1. M ∨ C
2. ¬D → ¬C
3. D → S
4. ¬M
5. C
6. D
7. S

Given
Given
Given
Given
DS (1,4)
MT (2,5)
MP (3,6)

Pat is smart!
Proof Techniques - direct proofs

A totally different example:
Prove that if $n$ is odd, then $5n + 3$ is even.
Before we prove it, we need to define even and odd.
How can we define an even number?

The integer $n$ is even if there exists an integer $k$ such that $n = 2k$

How can we define an odd number?

The integer $n$ is odd if there exists an integer $k$ such that $n = 2k + 1$
Proof Techniques - direct proofs

Prove that if n is odd, then $5n + 3$ is even.

Suppose n is odd,
Therefore $n = 2k + 1$ for some integer k.

Therefore $5n + 3 = 5(2k + 1) + 3$
= $10k + 5 + 3 = 10k + 8$
= $2(5k + 4)$ or $2(k')$
Example:
Prove that if \( n \) is odd, then \( n^2 \) is odd.
Proof Techniques - vacuous proofs

In general, to prove $p \rightarrow q$, assume $p$ and show that $q$ follows.

But $p \rightarrow q$ is also TRUE if $p$ is FALSE.

Ex. \quad p: x is odd
q: $x + 1$ is even

$\forall x, x \text{ odd} \rightarrow x+1 \text{ is even}$

what about when $x$ is 4?

Since $p$ is FALSE, $p \rightarrow q$ is TRUE
(but we don’t know a thing about $q$)
Proof Techniques - trivial proofs

In general, to prove $p \rightarrow q$, assume $p$ and show that $q$ follows.

But $p \rightarrow q$ is also TRUE if $q$ is TRUE.

Suggests proving $p \rightarrow q$ by proving $q$.

Ex. $p$: There is a Lion in the room
    $q$: $2 + 2 = 4$

Since $q$ is TRUE, $p \rightarrow q$ is TRUE
    (the truth or falsity of $p$ is irrelevant)
Recall that \( p \rightarrow q \equiv \neg q \rightarrow \neg p \) (the contrapositive)

So, we can prove the implication \( p \rightarrow q \) by first assuming \( \neg q \), and showing that \( \neg p \) follows.

Example: Given that \( a \) and \( b \) are integers,

Prove: if \( a + b \geq 15 \), then \( a \geq 8 \) or \( b \geq 8 \).

\[
(a + b \geq 15) \rightarrow (a \geq 8) \lor (b \geq 8)
\]

(Assume \( \neg q \)) Suppose \( (a < 8) \land (b < 8) \).

(Show \( \neg p \)) Then \( (a \leq 7) \land (b \leq 7) \),
and \( (a + b) \leq 14 \),
and \( (a + b) < 15 \). (\( \neg p \))
Proof Techniques - proof by contradiction

To prove a proposition \( p \), assume not \( p \) and show a contradiction.
(Prove that the sky is blue...Assume that the sky is not blue)

Suppose the proposition is of the form \( a \rightarrow b \), and recall that \( a \rightarrow b \equiv \neg a \vee b \equiv \neg (a \land \neg b) \). So assuming the opposite is to assume \( a \land \neg b \).

- For a conditional, we assume \( a \) and prove \( \neg b \)
- If I study hard, then I will earn an A
  - Assume I study hard and I will Not earn an A
Proof Techniques - proof by contradiction

Example:

Rainy days make gardens grow.
Gardens don’t grow if it is not hot.
When it is cold outside, it rains.

Prove that it’s (always) hot.

Given: $R \rightarrow G$
$\neg H \rightarrow \neg G$
$\neg H \rightarrow R$

Show: $H$

$((R \rightarrow G) \land (\neg H \rightarrow \neg G) \land (\neg H \rightarrow R)) \rightarrow H$?
Proof Techniques - proof by contradiction

Given: $R \rightarrow G$
   $\neg H \rightarrow \neg G$
   $\neg H \rightarrow R$

Show: $H$

1. $R \rightarrow G$  Given
2. $\neg H \rightarrow \neg G$  Given
3. $\neg H \rightarrow R$  Given
4. $\neg H$  assume to the contrary
5. $R$  MP (3,4)
6. $G$  MP (1,5)
7. $\neg G$  MP (2,4)
8. $G \land \neg G$  contradiction

$\therefore H$
Proof Techniques - proof by contradiction

Classic proof that $\sqrt{2}$ is irrational
- Irrational numbers are those that cannot be represented as a simple fraction

Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$ for some integers $a$ and $b$ (relatively prime)

Definition: $a$ and $b$ are relatively prime if they have no common factor other than 1

$\sqrt{2} = a/b$ implies

$2 = a^2/b^2$

$2b^2 = a^2$

$a^2$ is even, and so $a$ is even ($a = 2k$ for some $k$)

$2b^2 = (2k)^2 = 4k^2$

$b^2 = 2k^2$

$b^2$ is even, and so $b$ is even ($b = 2m$ for some $m$)

But if $a$ and $b$ are both even, then they are not relatively prime!
Contradiction!
Proof Techniques – switching back to contraposition

I claimed that if \( a^2 \) is even, then \( a \) is even, too.

To be complete, we should prove that, too.

Remember, to show \( p \rightarrow q \) by contraposition show \( \neg q \rightarrow \neg p \)

Show that if \( a \) is odd, then \( a^2 \) is odd

Then \( a = 2k + 1 \) for some integer \( k \)
Then \( a^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(j) + 1 \)
for some integer \( j \) and \( a^2 \) is odd

Therefore if \( a^2 \) is even, then \( a \) is even
Same idea with proof by contradiction

I claimed that if $a^2$ is even, then $a$ is even, too.

To be complete, we should prove that, too.

Remember, to show $p \rightarrow q$ by contradiction assume $p$ and $\neg q$ to be true

Suppose to the contrary $a^2$ is even, but $a$ is not.

Then $a = 2k + 1$ for some integer $k$
Then $a^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(j) + 1$ for some integer $j$ and $a^2$ is odd
But we know that $a^2$ is even.

So $a$ really is even.
Extra Examples
Group Exercise II

- Suppose you, the curious CSE 240 adventurer, finds a treasure map hidden inside the overly priced Discrete Mathematics Book

- You follow the map until it leads to a fork in the road

- Two helpful guides are at the fork in the road and willing to assist you

- One guide “Honest Abe” always tells the truth and will steer you to riches

- The other guide “Lying Lenny” always lies to you and does not want you to find the riches

- Unfortunately neither guide will tell you their name so you do not know who is who

- What SINGLE question can you ask either person to guarantee choosing the right path and finding the treasure?