Announcements

- Quiz 2 Today
- Exam in one week!

- Read Section 2.1 (Sets), 2.2 (Set Operations) and 5.1 (Mathematical Induction)
One More Example

• Show that at least 3 of any 25 days chosen must fall in the same month of the year
One More Example

• Show that at least 3 of any 25 days chosen must fall in the same month of the year

• Proof by contradiction
  – Suppose ¬ p
    • There are at most 2 days of any 25 days chosen that must fall in the same month of the year
    • Because there are 12 months in a year and 2 days could be chosen per month, we have a maximum of 24 days that could be selected. This contradicts choosing from 25 days. Therefore p]
Quick Background on Set Theory

A set is an unordered collection of elements.

Some examples:

{1, 2, 3} is the set containing “1” and “2” and “3.”
{1, 1, 2, 3, 3} = {1, 2, 3} since repetition is irrelevant.
{1, 2, 3} = {3, 2, 1} since sets are unordered.
{1, 2, 3, …} is a way we denote an infinite set (in this case, the natural numbers).
∅ = {} is the empty set, or the set containing no elements.

Note: ∅ ≠ {∅}
Set Theory - Definitions and notation

$x \in S$ means “$x$ is an element of set $S$.”

$x \notin S$ means “$x$ is not an element of set $S$.”

$A \subseteq B$ means “$A$ is a subset of $B$.”

or, “$B$ contains $A$.”

or, “every element of $A$ is also in $B$.”

or, $\forall x ((x \in A) \rightarrow (x \in B))$. 

Venn Diagram
Set Theory - Cardinality

If $S$ is finite, then the *cardinality* of $S$, $|S|$, is the number of distinct elements in $S$.

- If $S = \{1,2,3\}$, $|S| = 3$
- If $S = \{3,3,3,3,3\}$, $|S| = 1$
- If $S = \emptyset$, $|S| = 0$
- If $S = \{1, \{1\}, \{1,\{1\}\}\}$, $|S| = 3$
- If $S = \{0,1,2,3,...\}$, $|S|$ is infinity. (more on this later)
Set Theory - Power sets

If $S$ is a set, then the *power set* of $S$ is

$$2^S = \{ x : x \subseteq S \}.$$ 

If $S = \{a\}$,

$$P(S) = 2^S = \{\emptyset, \{a\}\}.$$ 

If $S = \{a, b\}$,

$$2^S = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}.$$ 

If $S = \emptyset$,

$$2^S = \{\emptyset\}.$$ 

If $S = \{\emptyset, \emptyset\}$,

$$2^S = \{\emptyset, \emptyset, \{\emptyset\}, \emptyset, \emptyset\\}.$$ 

Fact: if $S$ is finite, $|P(S)| = 2^{|S|}$. (if $|S| = n$, $|P(S)| = 2^n$)
Set Theory - Operators

The *union* of two sets $A$ and $B$ is:

$$A \cup B = \{ x : x \in A \lor x \in B \}$$

If $A = \{\text{Charlie, Lucy, Linus}\}$, and $B = \{\text{Lucy, Desi}\}$, then

$$A \cup B = \{\text{Charlie, Lucy, Linus, Desi}\}$$
On to Mathematical Induction

One rule: Due to peer pressure, if the person “before” you likes Lucky Charms, then you like Lucky Charms.

• Person 1 likes Lucky Charms.

• What can we conclude?

• Suppose we want to prove everyone likes Fruit Loops

• Need to show two things:

  • Person 1 likes Fruit Loops (FL(1))

  • If person k likes Fruit Loops, then person k+1 does too. (FL(k) → FL(k+1))
Mathematical Induction

• Suppose we want to prove everyone likes Fruit Loops
• Need to show two things:
  • Person 1 likes Fruit Loops (FL(1))
  • If person k likes Fruit Loops, then person k+1 does too. (FL(k) → FL(k+1))

• First part is a simple proposition we call the base case.
• Second part is a conditional. Start by assuming FL(k), and show that FL(k+1) follows.

∀n FL(n)

True by “peer pressure”
Mathematical Induction

Use induction to prove that the sum of the first $n$ positive odd integers is $n^2$.

Base case $(n=1)$: the sum of the first 1 odd integer is $1^2$. Yes, $1 = 1^2$.

Assume $P(k)$: the sum of the first $k$ odd integers is $k^2$. $1 + 3 + \ldots + (2k - 1) = k^2$

Prove that $1 + 3 + \ldots + (2k - 1) + (2k + 1) = (k+1)^2$

$1 + 3 + \ldots + (2k-1) + (2k+1) = k^2 + (2k + 1) = (k+1)^2$
Mathematical Induction

Prove that \( 1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1, \ \forall n \)

Base case (n=1): \( 1 \cdot 1! = (1+1)! - 1? \)
Yes, \( 1 \cdot 1! = 1, 2! - 1 = 1 \)

Assume P(k): \( 1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1 \)
Prove that \( 1 \cdot 1! + \ldots + k \cdot k! + (k+1)(k+1)! = (k+2)! - 1 \)

\[
1 \cdot 1! + \ldots + k \cdot k! + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!
= (k+1)(k+1)! + (k+1)! - 1
= (k+1)! ((k+1) + 1) - 1
= (k+1)!(k+2) - 1
= (k+2)! - 1
\]
Mathematical Induction

Prove that if a set $S$ has $|S| = n$, then $|P(S)| = 2^n$

- Base case ($n=0$): $S=\emptyset$, $P(S) = \{\emptyset\}$ and $|P(S)| = 1 = 2^0$

- Assume $P(k)$: If $|S| = k$, then $|P(S)| = 2^k$

- Prove that if $|S'| = k+1$, then $|P(S')| = 2^{k+1}$
  - $S' = S \cup \{a\}$ for some $S \subset S'$ with $|S| = k$, and $a \in S'$.
  - Partition the power set of $S'$ into the sets containing $a$ and those not.
  - We count these sets separately.
Mathematical Induction

- Assume $P(k)$: If $|S| = k$, then $|P(S)| = 2^k$

- Prove that if $|S'| = k+1$, then $|P(S')| = 2^{k+1}$
  
  - $S' = S \cup \{a\}$ for some $S \subseteq S'$ with $|S| = k$, and $a \in S'$.
  
  - Partition the power set of $S'$ into the sets containing $a$ and those not.

  - $P(S') = \{X : a \in X\} \cup \{X : a \not\in X\}$
  
  - $P(S') = \{X : a \in X\} \cup P(S)$

  Since these are all the subsets of elements in $S$.

  Subsets containing $a$ are made by taking any set from $P(S)$, and inserting an $a$. 

Mathematical Induction

- Assume $P(k)$: If $|S| = k$, then $|P(S)| = 2^k$

- Prove that if $|S'| = k+1$, then $|P(S')| = 2^{k+1}$

- $S' = S \cup \{a\}$ for some $S \subset S'$ with $|S| = k$, and $a \in S'$.

- $P(S') = \{X : a \in X\} \cup \{X : a \notin X\}$

- $P(S') = \{X : a \in X\} \cup P(S)$

- $|P(S')| = |\{X : a \in X\}| + |P(S)|$

  $= 2 \cdot |P(S)|$

  $= 2 \cdot 2^k = 2^{k+1}$
Group Activity – 2 “easy” problems

- If you have a 3 gallon jug and a five gallon jug, how could you put exactly four gallons into the five gallon jug?

- You have two strings whose only known property is that when you light one end of either string it takes exactly one hour to burn. The rate at which the strings will burn is completely random and each string is different.
  - How do you measure 45 minutes?