event that the bit string starts with a 1. Are $E$ and $F$ independent? If so, prove it. If not, explain why.

**Solution:** A bit string of length 3 contains an odd number of 1s if it has either one 1 or three 1s. The probability that it has one 1 is 3/8 (this is the binomial distribution with success probability 1/2). The probability that it has three 1s is 1/8. Thus, the probability that it has an odd number of 1s in 1/2, i.e., $\Pr(E) = 1/2$. The probability that it starts with a 1 is also clearly 1/2, i.e., $\Pr(F) = 1/2$. The probability that it starts with a 1 and has an odd number of 1s is exactly the probability of getting either 100 or 111. This is clearly 1/4 (each bit string has probability 1/8). That is, $\Pr(E \cap F) = 1/4$. Since $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$, it follows that $E$ and $F$ are independent.

14. [5 points] Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances. More precisely, define a sample space $S$, a probability $\Pr$ on $S$, and two random variables $X$ and $Y$ on $S$ such that $\operatorname{Var}(X + Y) \neq \operatorname{Var}(X) + \operatorname{Var}(Y)$. [Hint: make your life (and the graders’ lives) easier by taking the sample space to have two elements.]

**Solution:** There was a lot of variation in this answer, since any example was acceptable. What we were looking for is (a) a clear description of the sample space $S$, (c) a clear description of the random variables $X$ and $Y$, which showed that you understood that a random variable is a function from $S$ to the reals.) Here’s one example: Let $S = \{h, t\}$. Let $\Pr(h) = \Pr(t) = 1/2$. Define $X(h) = 1$, $X(t) = 0$, and $Y(h) = 0$ and $Y(t) = 1$. Note that $X + Y(s) = X + Y(t) = 1$. (Think of assigning toss of a coin. Count the number of times the coin is heads, and so it is either 0 or 1, depending on $X + Y$.)

$$\operatorname{Var}(X) = E((X - (E(X))^2) = E((X - 1/2)^2) = 1/2(0 - 1/2)^2 + 1/2(1 - 1/2)^2 = 1/4.$$  

Exactly the same reasoning shows that $\operatorname{Var}(Y) = 1/4$. Thus $\operatorname{Var}(X) + \operatorname{Var}(Y) = 1/2$. But $X + Y$ is a constant function, so $\operatorname{Var}(X + Y) = 0$. Clearly, $\operatorname{Var}(X) + \operatorname{Var}(Y) \neq \operatorname{Var}(X + Y)$.

15. [4 points] Are the formulas $p \Rightarrow (q \Rightarrow r)$ and $(p \Rightarrow q) \Rightarrow r$ equivalent? (If you think they are, then show that their truth tables are identical. If you think they are not, give a truth assignment that shows they are different.)

**Solution:** These formulas are not equivalent. This is probably easiest to see if we recall that $A \Rightarrow B$ is equivalent to $\neg A \lor B$. Thus, $p \Rightarrow (q \Rightarrow r)$ is equivalent to $\neg p \lor \neg q \lor r$, and $(p \Rightarrow q) \Rightarrow r$ is equivalent to $(\neg p \lor q) \lor r$, which is equivalent to $(p \land \neg q) \lor r$, which is equivalent to $(p \land \neg q) \lor r$. Consider the truth assignment that makes $p$ false, $q$ true, and $r$ false. Under this truth assignment, $p \Rightarrow (q \Rightarrow r)$ is true, but $(p \Rightarrow q) \Rightarrow r$ is false. (You could also consider the truth assignment that makes $p$ true, $q$ false, and $r$ false; it gives the same result.)

16. [6 points] Which of the following formulas is true if the domain is the natural numbers, and which are true if the domain is the real numbers. (Explain your answer in each case.)

(a) $\exists x \forall y (x < y \land \forall z (z \leq x \lor y \leq z))$