Announcements

• Exam 1 will be passed back next Tuesday
  – We will go over the exam in class

• Quiz 3 is today

• Quiz 4 in one week

• Homework 4 is due on Thursday

• Read Section 2.1 (Sets), 2.2 (Set Operations) and 5.1 (Mathematical Induction) by Thursday
Set Theory (Review)

A set is an unordered collection of elements.

Some examples:

\{1, 2, 3\} is the set containing “1” and “2” and “3.”
\{1, 1, 2, 3, 3\} = \{1, 2, 3\} since repetition is irrelevant.
\{1, 2, 3\} = \{3, 2, 1\} since sets are unordered.
\{1, 2, 3, \ldots\} is a way we denote an infinite set (in this case, the natural numbers).
\emptyset = {} is the empty set, or the set containing no elements.

Note: \emptyset \neq \{\emptyset\}
Set Theory - Definitions and notation

$x \in S$ means “$x$ is an element of set $S$."

$x \notin S$ means “$x$ is not an element of set $S$."

$A \subseteq B$ means “$A$ is a subset of $B$."

or, “$B$ contains $A$."

or, “every element of $A$ is also in $B$."

or, $\forall x ((x \in A) \rightarrow (x \in B))$.

Venn Diagram
Set Theory - Definitions and notation

A ⊆ B means “A is a subset of B.”

A = B if and only if A and B have exactly the same elements.

iff, A ⊆ B and B ⊆ A
iff, ∀x ((x ∈ A) ↔ (x ∈ B)).

So to show equality of sets A and B, show:
A ⊆ B
B ⊆ A
Set Theory - Definitions and notation

A ⊂ B means “A is a proper subset of B.”

- A ⊆ B, and A ≠ B.
- ∀x ((x ∈ A) → (x ∈ B)) ∧ ¬∀x ((x ∈ B) → (x ∈ A))
- ∀x ((x ∈ A) → (x ∈ B)) ∧ ∃x ((x ∈ B) ∧ ¬(x ∈ A))
Set Theory - Definitions and notation

Quick examples:
\{1,2,3\} \subseteq \{1,2,3,4,5\}
\{1,2,3\} \subset \{1,2,3,4,5\}

Is \emptyset \subseteq \{1,2,3\}?
Yes! \forall x (x \in \emptyset) \rightarrow (x \in \{1,2,3\}) holds, because (x \in \emptyset) is false.

Is \emptyset \in \{1,2,3\}?
No!

Is \emptyset \subseteq \{\emptyset,1,2,3\}?
Yes!

Is \emptyset \in \{\emptyset,1,2,3\}?
Yes!
Set Theory - Definitions and notation

Quiz time:

Is \( x \subseteq \{x\} \)?  
No

Is \( \{x\} \subseteq \{x\} \)?  
Yes

Is \( \{x\} \in \{x,\{x\}\} \)?  
Yes

Is \( \{x\} \subseteq \{x,\{x\}\} \)?  
Yes

Is \( \{x\} \in \{x\} \)?  
No
Set Theory - Ways to define sets

- Explicitly: \{John, Paul, George, Ringo\}
- Implicitly: \{1,2,3,…\}, or \{2,3,5,7,11,13,17,…\}
- Set builder: \{ x : x is prime \}, \{ x | x is odd \}.
  In general \{ x : P(x) is true \}, where P(x) is some description of the set.

: and | are read “such that” or “where”

Ex. Let \(D(x,y)\) denote “\(x\) is divisible by \(y\).”
Give another name for
\[
\{ x : \forall y ((y > 1) \land (y < x)) \rightarrow \neg D(x,y) \}.
\]

What is this set of numbers?
Primes
If $S$ is finite, then the *cardinality* of $S$, $|S|$, is the number of distinct elements in $S$.

- If $S = \{1,2,3\}$, $|S| = 3$.
- If $S = \{3,3,3,3,3\}$, $|S| = 1$.
- If $S = \emptyset$, $|S| = 0$.
- If $S = \{1, \{1\}, \{1,\{1\}\}\}$, $|S| = 3$.
- If $S = \{0,1,2,3,\ldots\}$, $|S|$ is (one kind of) infinity. (more on this later)
If $S$ is a set, then the power set of $S$ is

$$2^S = \{ x : x \subseteq S \}.$$

If $S = \{a\}$,

$$2^S = \{\{\}, \{a\}\}.$$  

If $S = \{a,b\}$,

$$2^S = \{\{\}, \{a\}, \{b\}, \{a,b\}\}.$$  

If $S = \emptyset$,

$$2^S = \{\}.$$  

If $S = \{\emptyset, \emptyset\}$,

$$2^S = \{\{\}, \emptyset, \{\emptyset\}, \emptyset, \{\emptyset\}\}.$$  

Fact: if $S$ is finite, $|P(S)| = 2^{|S|}$. (If $|S| = n$, $|P(S)| = 2^n$)
The *Cartesian Product* of two sets $A$ and $B$ is:

$$A \times B = \{ <a,b> : a \in A \land b \in B \}$$

If $A = \{ \text{Charlie, Lucy, Linus} \}$, and $B = \{ \text{Brown, VanPelt} \}$, then

$$A \times B = \{ <\text{Charlie, Brown}>, <\text{Charlie, VanPelt}>, <\text{Lucy, Brown}>, <\text{Lucy, VanPelt}>, <\text{Linus, Brown}>, <\text{Linus, VanPelt}> \}$$

$$A_1 \times A_2 \times \ldots \times A_n = \{ <a_1, a_2, \ldots, a_n> : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \}$$
Set Theory - Operators

The union of two sets $A$ and $B$ is:

$$A \cup B = \{ x : x \in A \lor x \in B \}$$

If $A = \{\text{Charlie, Lucy, Linus}\}$, and $B = \{\text{Lucy, Desi}\}$, then

$$A \cup B = \{\text{Charlie, Lucy, Linus, Desi}\}$$
Set Theory - Operators

The *intersection* of two sets $A$ and $B$ is:

$A \cap B = \{ x : x \in A \land x \in B \}$

If $A = \{\text{Charlie, Lucy, Linus}\}$, and
$B = \{\text{Lucy, Desi}\}$, then

$A \cap B = \{\text{Lucy}\}$
Another example

If $A = \{x : x \text{ is a US president}\}$, and
$B = \{x : x \text{ is deceased}\}$, then

$A \cap B = \{x : x \text{ is a deceased US president}\}$
Set Theory - Operators

One more example

If $A = \{x : x \text{ is a US president}\}$, and $B = \{x : x \text{ is in this room}\}$, then

$$A \cap B = \{x : x \text{ is a US president in this room}\} = \emptyset$$

Sets whose intersection is empty are called *disjoint* sets.
Set Theory - Operators

The complement of a set $A$ is:

$\overline{A} = \{ x : x \notin A \}$

If $A = \{x : x \text{ is bored}\}$, then $A = \{x : x \text{ is not bored}\}$

$\emptyset = \bigcup$ and $\bigcup = \emptyset$
Set Theory - Operators

The set difference, $A - B$, is:

$A - B = \{ x : x \in A \land x \notin B \}$

$A - B = A \cap B$
Set Representation

• **How could you represent a set on a computer?**
  – Bitmap Representation
    • The set of 10 numbers, with 1, 3, and 5 set
      – 1010100000
  – Linked List

```
  1 -> 3 -> 5
```

• **How would you complement the set?**
  – Bitmap?
  – Linked List?
The symmetric difference, $A \oplus B$, is:

$$A \oplus B = \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \}$$

$$= (A - B) \cup (B - A)$$

like “exclusive or”
Set Theory - Operators

\[ A \oplus B = \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \} \]

\[ = (A - B) \cup (B - A) \]

Proof:

\[ \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \} \]

\[ = \{ x : (x \in A - B) \lor (x \in B - A) \} \]

\[ = \{ x : x \in ((A - B) \cup (B - A)) \} \]

\[ = (A - B) \cup (B - A) \]
Set Theory - Famous Identities

Identity

\[ A \cap U = A \]
\[ A \cup \emptyset = A \]

Domination

\[ A \cup U = U \]
\[ A \cap \emptyset = \emptyset \]

Idempotent

\[ A \cup A = A \]
\[ A \cap A = A \]
Set Theory - Famous Identities

- **Excluded Middle** \( A \cup \bar{A} = U \)

- **Uniqueness** \( A \cap \bar{A} = \emptyset \)

- **Double complement** \( A = \bar{A} \)
Set Theory - Famous Identities

- **Commutativity**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$

- **Associativity**
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$

- **Distributivity**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Set Theory - Famous Identities

- DeMorgan’s I
  
  
  \( (A \cup B) = A \cap B \)

- DeMorgan’s II
  
  
  \( (A \cap B) = A \cup B \)

Hand waving is good for intuition, but we aim for a more formal proof.
Set Theory – 4 Ways to prove identities

• Show that $A \subseteq B$ and that $B \subseteq A$.
• Use a membership table.
• Use previously proven identities.
• Use logical equivalences to prove equivalent set definitions.

Like truth tables
Like $\equiv$
Not hard, a little tedious
Set Theory – 4 Ways to prove identities

Prove that \((A \cup B) = A \cap B\)

1. \((\subseteq)\) \((x \in A \cup B) \rightarrow (x \notin A \cup B) \rightarrow\)
   \((x \notin A \text{ and } x \notin B) \rightarrow (x \in A \cap B)\)

2. \((\supseteq)\) \((x \in A \cap B) \rightarrow (x \notin A \text{ and } x \notin B)\)
   \(\rightarrow (x \notin A \cup B) \rightarrow (x \in A \cup B)\)
Set Theory – 4 Ways to prove identities

Prove that \((A \cup B) = A \cap B\) using a membership table.

0 : x is not in the specified set
1 : otherwise

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(\overline{A})</th>
<th>(\overline{B})</th>
<th>(A \cap B)</th>
<th>(A \cup B)</th>
<th>(\overline{A \cup B})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
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</tbody>
</table>

Haven't we seen this before?
Set Theory – 4 Ways to prove identities

Prove that \((A \cup B) = A \cap B\) using logically equivalent set definitions.

\[ (A \cup B) = \{x : \neg(x \in A \lor x \in B)\} \]

\[ = \{x : \neg(x \in A) \land \neg(x \in B)\} \]

\[ = \{x : (x \in A) \land (x \in B)\} \]

\[ = A \cap B \]
Set Theory - Generalized Union

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k > 1, x = ki, k \in \mathbb{N} \} \]

\[ A_1 = \{2,3,4,\ldots\} \]
\[ A_2 = \{4,6,8,\ldots\} \]
\[ A_3 = \{6,9,12,\ldots\} \]
Set Theory - Generalized Union

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k > 1, x = ki, k \in \mathbb{N} \} \]

Then

\[ \bigcup_{i=2}^{\infty} A_i = ? \]

primes

\[ \begin{align*}
a) & \text{ Primes} \\
b) & \text{ Composites} \\
c) & \emptyset \\
d) & \mathbb{N} \\
e) & \text{ I have no clue.} \\
\end{align*} \]
Set Theory - Generalized Intersection

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k, x = ki, k \in \mathbb{N} \} \]

\( A_1 = \{1,2,3,4,\ldots\} \)
\( A_2 = \{2,4,6,\ldots\} \)
\( A_3 = \{3,6,9,\ldots\} \)
Set Theory - Generalized Intersection

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k, x = ki, k \in \mathbb{N} \} \]

Then

\[ \bigcap_{i=1}^{n} A_i = \ ? \quad \text{Multiples of } LCM(1,\ldots,n) \]
Set Theory - Inclusion/Exclusion

Example:
How many people are wearing a watch?
How many people are wearing sneakers?

How many people are wearing a watch OR sneakers?

\[ |A \cup B| = |A| + |B| - |A \cap B| \]
Set Theory - Inclusion/Exclusion

Example:
There are 83 cs majors.
40 are taking cs240.
31 are taking cs101.
22 are taking both.

How many are taking neither?

\[ 83 - (40 + 31 - 22) = 34 \]
Set Theory - Generalized Inclusion/Exclusion

Suppose we have:

And I want to know $|A \cup B \cup C|$

$$|A \cup B \cup C| = |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$
Quiz