One of the problems below will be chosen at random in class for a quiz.

1. Use strong induction to show that every positive integer \( n \) can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers \( 2^0 = 1, 2^1 = 2, 2^2 = 4 \), and so on. Note each power of two can only be used once. [Hint: For the inductive step, separately consider the case where \( k + 1 \) is even and where it is odd. When \( k + 1 \) is even, note that \( (k + 1)/2 \) is an integer.]

2. Suppose you begin with a pile of \( n \) stones and split this pile into \( n \) piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have \( r \) and \( s \) stones in them, respectively, you compute \( rs \). Show that no matter how you split the piles, the sum of the products computed at each step equals \( n(n - 1)/2 \).

3. Find the least integer \( n \) such that \( f(x) \) is \( O(x^n) \) for each of the following functions.
   
   (a) \( f(x) = 2x^2 + x^3 \log x \)
   
   (b) \( f(x) = 3x^5 + (\log x)^4 \)
   
   (c) \( f(x) = (x^4 + x^2 + 1)/(x^4 + 1) \)
   
   (d) \( f(x) = (x^3 + 5 \log x)/(x^4 + 1) \)

4. Find the flaw with the following “proof” that \( a^n = 1 \) for all nonnegative integers \( n \), whenever \( a \) is a nonzero real number.

   **Basis Step:** \( a^0 = 1 \) is true by definition of \( a^0 \).
   
   **Inductive Step:** Assume that \( a^j = 1 \) for all nonnegative integers \( j \) with \( j \leq k \). Then note that \( a^{k+1} = \frac{a^k \cdot a^k}{a^k} = \frac{1 \cdot 1}{1} = 1 \)

5. Prove that 6 divides \( n^3 - n \) whenever \( n \) is a nonnegative integer.

6. Prove that if \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) are sets such that \( A_j \subseteq B_j \) for all \( j \in [n] \) then \( \bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j \).