Abstract

We study optimal control problems of spiking neurons whose dynamics are described by a phase model. We design minimum-power current stimuli (controls) that lead to targeted spiking times. In particular, we consider bounded control amplitude and characterize the range of possible spiking times determined by the bound which can be chosen sufficiently small within the range that the phase model is valid. We show that for a given bound, the corresponding feasible spiking times are optimally achieved by piecewise continuous controls. We present analytic expressions of these minimum-power stimuli with numerical simulations for several popular phase models.

Introduction

Control of neurons and hence the nervous system by external current stimuli (controls) has received increased scientific attention in recent years for its wide range of applications from deep brain stimulation to oscillatory neurocomputers. We represent the neuron oscillators by phase-reduced models, which form a standard nonlinear system, and the minimum-power stimuli for complete cycle of phase rotation is derived from Pontryagin’s Maximum Principle.

The design of such minimum-power stimuli to elicit spikes of neuron oscillators has clinical importance, notably in deep brain stimulation therapy for Parkinson’s disease and essential tremor, where mild stimulations are required. In addition, interest of reducing the energy consumption in neurological implants such as cardiac pacemakers makes such optimal designs imperative.

Phase Reduce Models for Neural Oscillators

A periodically spiking or firing neuron can be reduced to a periodic oscillator governed by the nonlinear dynamical equation of the form,

\[
\frac{d\theta}{dt} = f(\theta) + Z(\theta)I(t),
\]

where \( \theta \) is Phase, \( t \) is Time, \( I(t) \) is Current Stimulus (Control), \( T_1 \) is Targeted Spiking Time, \( t(t) \) is Baseline Dynamics, \( Z(\theta) \) is Phase Response Curve.

The assumption that \( Z(\theta) \) vanishes only on isolated points and that \( f(\theta) > 0 \) are made so that a full revolution of the phase is possible. By convention, neuron spikes at \( \theta = 2n\pi \), where \( n \in \mathbb{N} \), e.g., \( \theta = 0 \) or \( 2\pi \). In the absence of any input \( I(t) \), the neuron spikes periodically at its natural frequency, while the spiking time can be advanced or delayed in a desired manner by an appropriate choice of \( I(t) \).

Optimal Control Problem

\[
\min_{I(t)} \int_0^{T_1} |I(t)|^2 \, dt
\]

s.t. \( \dot{\theta} = f(\theta) + Z(\theta)I(t), \)

\( \theta(0) = 0, \quad \theta(T_1) = 2\pi \)

where \( T_1 \) is Targeted Spiking Time, \( M \) is Current Amplitude Bound.

Hamiltonian

\[
H = \dot{\theta}^2 + \lambda(\omega + z_\theta \sin \theta - I)
\]

Optimal Control for Sinusoidal Phase Model

Sinusoidal phase model is given by,

\[
\dot{\theta} = \omega + z_\theta \sin \theta \cdot I(t),
\]

where \( \omega \) is the natural oscillation frequency of the neuron and \( z_\theta \) is a model-dependent constant. Note that this type of PRC’s can be obtained by periodic orbits near the super critical Hopf bifurcation. This type of bifurcation occurs for Type II neuron models like Fitzhugh-Nagumo model. The optimal control without the amplitude constraint is given by,

\[
I^* = \frac{-\omega + \sqrt{\omega^2 - \omega^2 z_\theta^2 \sin^2 \theta}}{z_\theta \sin \theta},
\]

where,

\[
T_1 = \int_0^{2\pi} \frac{1}{\sqrt{\omega^2 - \omega^2 z_\theta^2 \sin^2 \theta}} \, d\theta.
\]

With the control amplitude constraint the feasible spiking time is limited and the optimal control has to be chosen according to the spiking time. \( T_1 \) is Minimum possible spiking time, \( T^* \) is Maximum possible spiking time, \( T^*_1 \) is Minimum spiking time by control \( I^* \), \( T^*_2 \) is Maximum spiking time by control \( I^* \).

\[\begin{align*}
I_1 &= \begin{cases} I^* & 0 \leq \theta < \theta_1 \\
M & \theta_1 \leq \theta \leq \theta_2 \\
- M & \theta_2 \leq \theta \leq \theta_3 \\
M & \theta_3 \leq \theta \leq \theta_4 \\
M & \theta_4 \leq \theta < 2\pi,
\end{cases} \\
I_2 &= \begin{cases} I^* & 0 \leq \theta < \theta_5 \\
M & \theta_5 \leq \theta \leq \theta_6 \\
M & \theta_6 \leq \theta < \theta_7 \\
M & \theta_7 \leq \theta < \theta_8 \\
M & \theta_8 \leq \theta < 2\pi,
\end{cases}
\end{align*}\]

where,

\[
\theta_1 = \sin^{-1}(1 - 2M/(z_\theta M^2 \cdot \omega \lambda_0)),
\]
\[
\theta_2 = \pi - \theta_1,
\]
\[
\theta_3 = \pi + \theta_1,
\]
\[
\theta_4 = 2\pi - \theta_1,
\]
\[
\lambda_0 = \frac{\omega}{\omega - 1},
\]
\[
\lambda_5 = \theta_0,
\]
\[
\lambda_6 = \frac{\omega}{\omega - 1},
\]
\[
\lambda_7 = \frac{\omega}{\omega - 1},
\]
\[
\lambda_8 = \frac{\omega}{\omega - 1},
\]
\[
\lambda_9 = \frac{\omega}{\omega - 1},
\]

The optimal control selection scenarios are depicted below.

Optimal Control For Morris-Lecar PRC

Many of experimentally determined PRC’s for real neurons have arbitrary shapes. Therefore, we apply the derived optimal control strategies to the Morris-Lecar PRC which obtained numerically by Morris-Lecar neuron model. Previous work has shown that the PRC for an Aplysia motoneuron, which can be experimentally observed, is extremely similar to that of a Morris-Lecar neuron. As a result, we find minimum-power stimuli for Morris-Lecar PRC to demonstrate the applicability and generality of our analytic method to practical PRC’s.

The phase model of the Morris-Lecar neuron can be written as,

\[
\dot{\theta} = \omega + Z(\theta)I(t),
\]

where the PRC is given by \( Z(\theta) \). It has the period \( T = 22.211 \) ms and natural frequency \( \omega = 0.283 \) rad/ms for some standard set of parameter values which we used for this calculation. The spiking range with control amplitude bound \( M = 0.01 \mu A \) is \([9.623, 26.288] \) ms. Therefore we can design the optimal controls to spike this neuron in this interval with minimum power.

Discussion

Minimum-power stimuli for steering any nonlinear phase reduced oscillators between desired initial and target states can be derived following the method presented here. In addition, the charge-balanced constraint can be readily incorporated into this framework, which is of clinical importance in deep brain stimulations.

The optimal control of a single neuron system investigated in this work illustrates the fundamental limit of spiking a neuron with external stimuli and provides a benchmark structure that enables us to study optimal control of spiking neuron populations. Our recent work showed that simultaneously spiking a network of neurons with weak forcing is possible. However, many of the related optimal control problems such as minimum-power or time-optimal controls for firing a neural network have not been studied yet.

References