

Comparing Systems Using Sample Data



- ❑ Sample Versus Population
- ❑ Confidence Interval for The Mean
- ❑ Approximate Visual Test
- ❑ One Sided Confidence Intervals
- ❑ Confidence Intervals for Proportions
- ❑ Sample Size for Determining Mean and proportions

Sample

- ❑ Old French word `essample'
⇒ `sample' and `example'
- ❑ One example \neq theory
- ❑ One sample \neq Definite statement

Sample Versus Population

- Generate several million random numbers with mean μ and standard deviation σ

Draw a sample of n observations

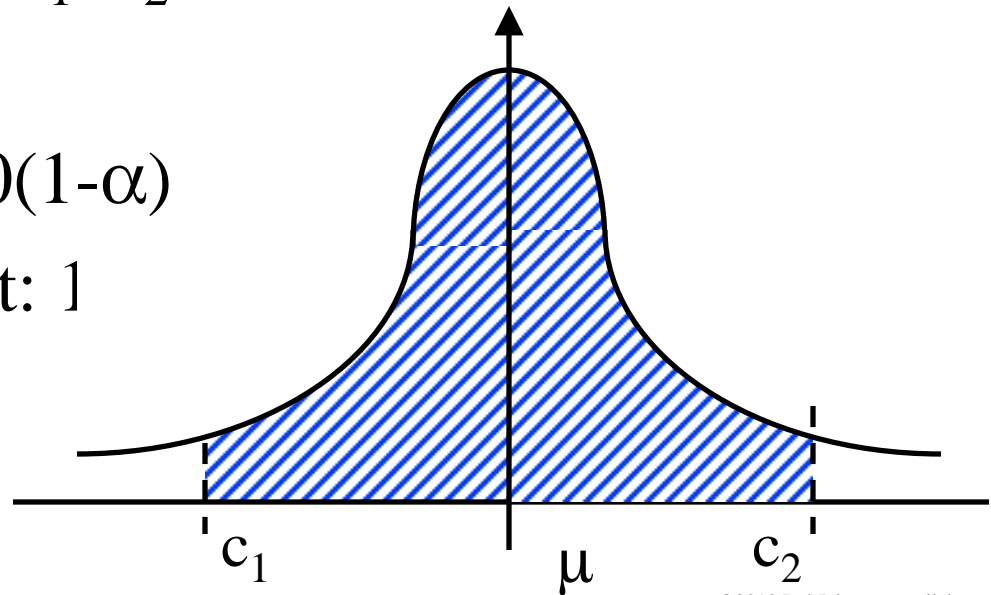
$$\bar{x} \neq \mu$$

- Sample mean \neq population mean
- Parameters: population characteristics
= Unknown = Greek
- Statistics: Sample estimates = Random = English

Confidence Interval for The Mean

- ❑ k samples $\Rightarrow k$ Sample means
 \Rightarrow Can't get a single estimate of μ
 \Rightarrow Use bounds $c_{\{1\}}$ and $c_{\{2\}}$:
Probability $\{c_1 \leq \mu \leq c_2\} = 1 - \alpha$

- ❑ Confidence interval: $[(c_1, c_2)]$
- ❑ Significance level: α
- ❑ Confidence level: $100(1-\alpha)$
- ❑ Confidence coefficient: 1



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Determining Confidence Interval

- ❑ Use 5-percentile and 95-percentile of the sample means to get 90% Confidence interval \Rightarrow Need many samples.
- ❑ Central limit theorem: Sample mean of independent and identically distributed observations:

$$\bar{x} \sim N(\mu, \sigma / \sqrt{n})$$

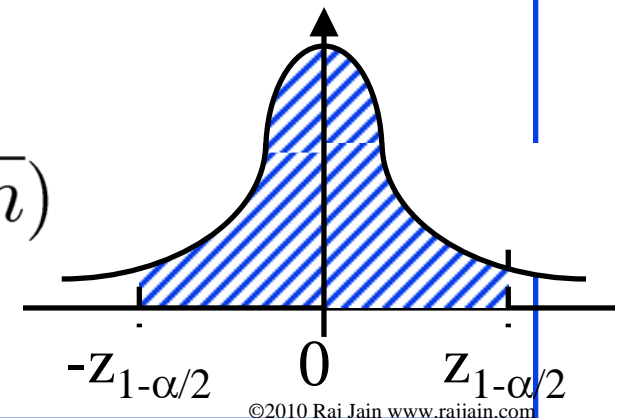
Where μ = population mean, σ = population standard deviation

- ❑ Standard Error: Standard deviation of the sample mean
 $= \sigma / \sqrt{n}$

- ❑ 100(1-a)% confidence interval for μ :

$$(\bar{x} - z_{1-\alpha/2} s / \sqrt{n}, \bar{x} + z_{1-\alpha/2} s / \sqrt{n})$$

$$z_{1-\alpha/2} = (1-\alpha/2)\text{-quantile of } N(0,1)$$



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Example 13.1

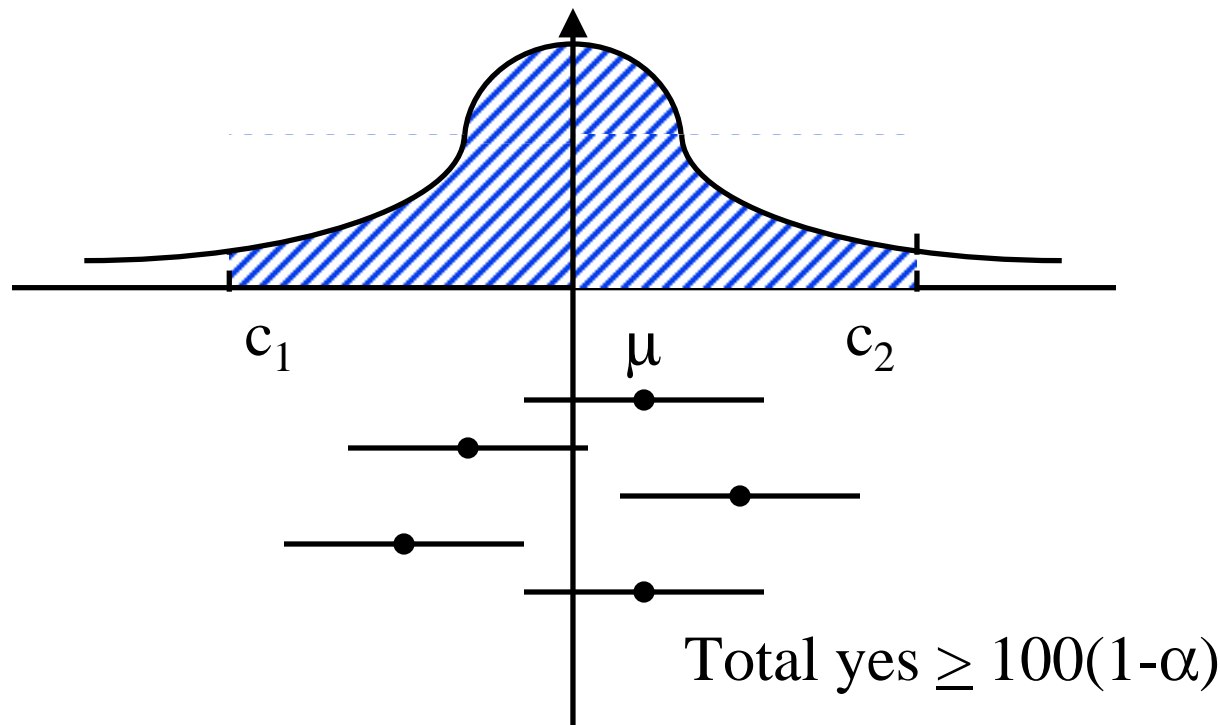
- $\bar{x} = 3.90$, $s = 0.95$ and $n = 32$
- A 90% confidence interval for the mean
 $= 3.90 \mp (1.645)(0.95)/\sqrt{32} = (3.62, 4.17)$
- We can state with 90% confidence that the population mean is between 3.62 and 4.17. The chance of error in this statement is 10%.

A 95% confidence interval for the mean $= 3.90 \mp (1.960)(0.95)/\sqrt{32}$
 $= (3.57, 4.23)$

A 99% confidence interval for the mean $= 3.90 \mp (2.576)(0.95)/\sqrt{32}$
 $= (3.46, 4.33)$

Confidence Interval: Meaning

- If we take 100 samples and construct confidence interval for each sample, the interval would include the population mean in 90 cases.



Confidence Interval for Small Samples

- 100(1- α) % confidence interval for for $n < 30$:

$$(\bar{x} - t_{[1-\alpha/2; n-1]} s / \sqrt{n}, \bar{x} + t_{[1-\alpha/2; n-1]} s / \sqrt{n})$$

- $t_{[1-\alpha/2; n-1]} = (1-\alpha/2)$ -quantile of a t-variate with $n-1$ degrees of freedom

$$x \sim N(\mu, \sigma^2)$$

$$\Rightarrow (\bar{x} - \mu) / (\sigma / \sqrt{n}) \sim N(0, 1)$$

$$(n - 1) s^2 / \sigma^2 \sim \chi^2(n - 1)$$

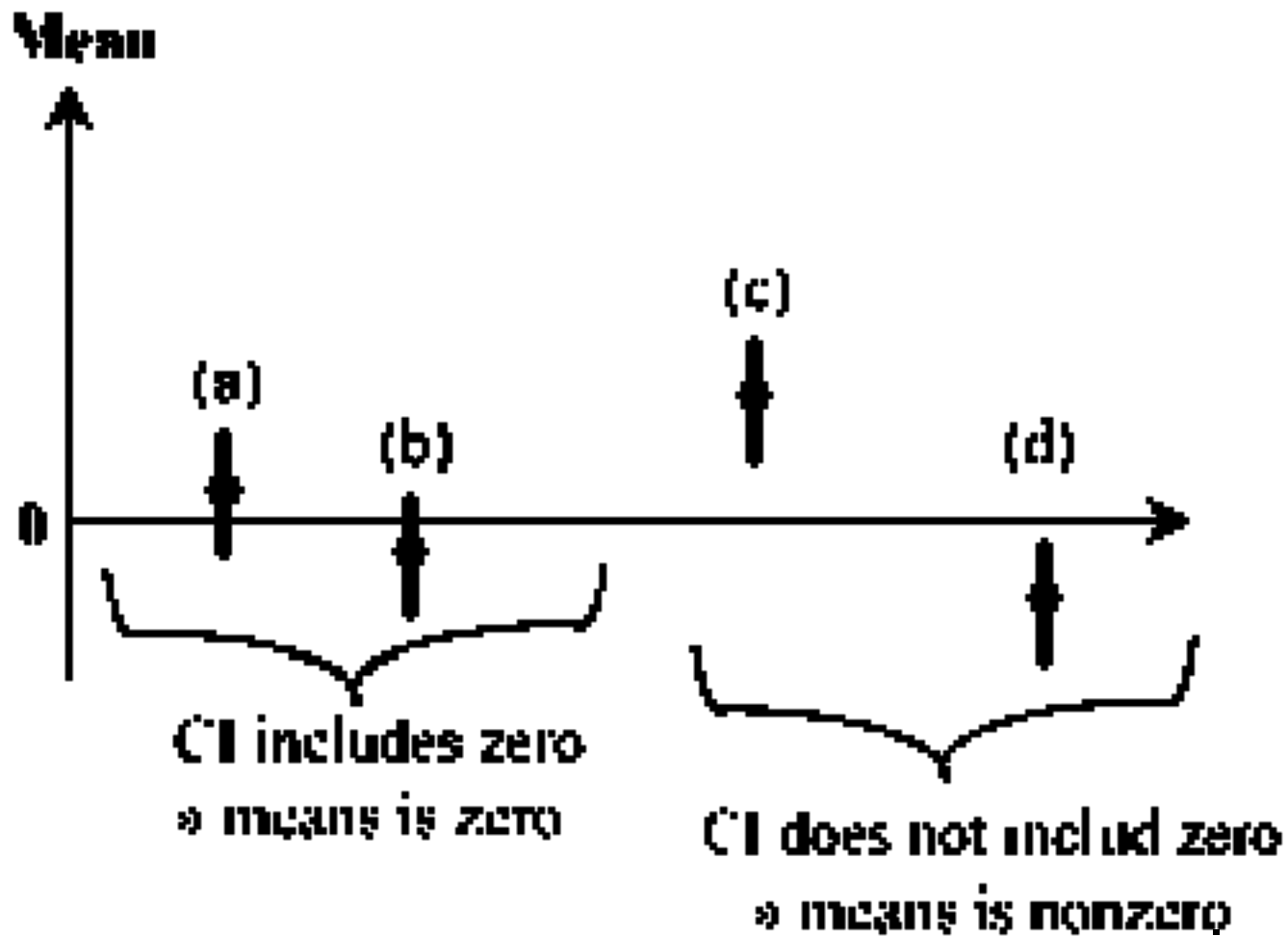
$$(\bar{x} - \mu) / \sqrt{s^2 / n} \sim t(n - 1)$$

Example 13.2

- ❑ Sample: -0.04, -0.19, 0.14, -0.09, -0.14, 0.19, 0.04, and 0.09.
- ❑ Mean = 0, Sample standard deviation = 0.138.
- ❑ For 90% interval: $t_{[0.95;7]} = 1.895$
- ❑ Confidence interval for the mean

$$0 \mp 1.895 \times 0.138 = 0 \mp 0.262 = (-0.262, 0.262)$$

Testing For A Zero Mean



Example 13.3

- Difference in processor times: {1.5, 2.6, -1.8, 1.3, -0.5, 1.7, 2.4}.
- Question: Can we say with 99% confidence that one is superior to the other?

$$\text{Sample size} = n = 7$$

$$\text{Mean} = 7.20/7 = 1.03$$

$$\text{Sample variance} = (22.84 - 7.20 \cdot 7.20/7)/6 = 2.57$$

$$\text{Sample standard deviation} = \sqrt{2.57} = 1.60$$

$$\text{Confidence interval} = 1.03 \mp t * 1.60/\sqrt{7} = 1.03 \mp 0.6t$$

$$100(1 - \alpha) = 99, \alpha = 0.01, 1 - \alpha/2 = 0.995$$

$$t_{[0.995; 6]} = 3.707$$

- 99% confidence interval = (-1.21, 3.27)

Example 13.3 (Cont)

- ❑ Opposite signs \Rightarrow we cannot say with 99% confidence that the mean difference is significantly different from zero.
- ❑ Answer: They are same.
- ❑ Answer: The difference is zero.

Example 13.4

- ❑ Difference in processor times: {1.5, 2.6, -1.8, 1.3, -0.5, 1.7, 2.4}.
- ❑ Question: Is the difference 1?
- ❑ 99% Confidence interval = (-1.21, 3.27)
- ❑ Yes: The difference is 1

Paired vs. Unpaired Comparisons

- ❑ **Paired**: one-to-one correspondence between the i th test of system A and the i th test on system B
- ❑ Example: Performance on i th workload
- ❑ Use confidence interval of the difference
- ❑ **Unpaired**: No correspondence
- ❑ Example: n people on System A, n on System B
⇒ Need more sophisticated method

Example 13.5

- ❑ Performance: $\{(5.4, 19.1), (16.6, 3.5), (0.6, 3.4), (1.4, 2.5), (0.6, 3.6), (7.3, 1.7)\}$. Is one system better?
- ❑ Differences: $\{-13.7, 13.1, -2.8, -1.1, -3.0, 5.6\}$.

Sample mean $= -0.32$

Sample variance $= 81.62$

Sample standard deviation $= 9.03$

Confidence interval for the mean $= -0.32 \pm t\sqrt{(81.62/6)}$
 $= -0.32 \pm t(3.69)$

$t_{[0.95,5]} = 2.015$

90% confidence interval $= -0.32 \pm (2.015)(3.69)$
 $= (-7.75, 7.11)$

- ❑ Answer: No. They are not different.

Unpaired Observations

- Compute the sample means:

$$\bar{x}_a = \frac{1}{n_a} \sum_{i=1}^{n_a} x_{ia}$$

$$\bar{x}_b = \frac{1}{n_b} \sum_{i=1}^{n_b} x_{ib}$$

- Compute the sample standard deviations:

$$s_a = \left\{ \frac{(\sum_{i=1}^{n_a} x_{ia}^2) - n_a \bar{x}_a^2}{n_a - 1} \right\}^{\frac{1}{2}}$$

$$s_b = \left\{ \frac{(\sum_{i=1}^{n_b} x_{ib}^2) - n_b \bar{x}_b^2}{n_b - 1} \right\}^{\frac{1}{2}}$$

Unpaired Observations (Cont)

- Compute the mean difference: $(\bar{x}_a - \bar{x}_b)$
- Compute the standard deviation of the mean difference:

$$s = \sqrt{\left(\frac{s_a^2}{n_a} + \frac{s_b^2}{n_b} \right)}$$

- Compute the effective number of degrees of freedom:

$$\nu = \frac{\left(\frac{s_a^2}{n_a} + \frac{s_b^2}{n_b} \right)^2}{\frac{1}{n_a+1} \left(\frac{s_a^2}{n_a} \right)^2 + \frac{1}{n_b+1} \left(\frac{s_b^2}{n_b} \right)^2} - 2$$

- Compute the confidence interval for the mean difference:

$$(\bar{x}_a - \bar{x}_b) \mp t_{[1-\alpha/2; \nu]} s$$

Example 13.6

□ Times on System A: {5.36, 16.57, 0.62, 1.41, 0.64, 7.26}

Times on system B: {19.12, 3.52, 3.38, 2.50, 3.60, 1.74}

□ Question: Are the two systems significantly different?

□ For system A:

$$\text{Mean } \bar{x}_a = 5.31$$

$$\text{Variance } s_a^2 = 37.92$$

$$n_a = 6$$

□ For System B:

$$\text{Mean } \bar{x}_b = 5.64$$

$$\text{Variance } s_b^2 = 44.11$$

$$n_b = 6$$

Example 13.6 (Cont)

Mean difference $\bar{x}_a - \bar{x}_b = -0.33$

Standard deviation of the mean difference = 3.698

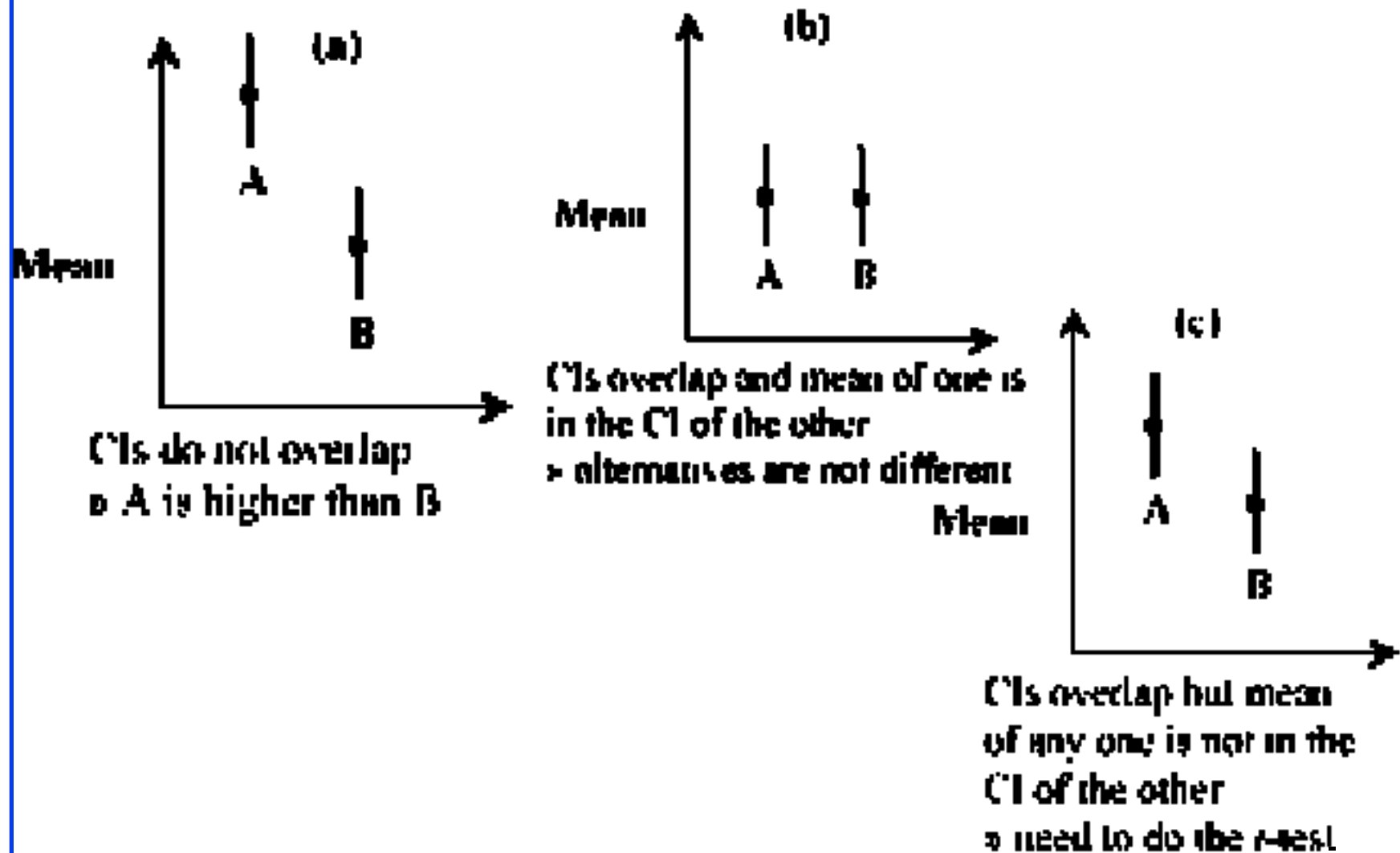
Effective number of degrees of freedom $f = 11.921$

The 0.95-quantile of a t-variate with 12 degrees of freedom = 1.71

The 90% confidence interval for the difference = $(-6.92, 6.26)$

- The confidence interval includes zero
 \Rightarrow the two systems are not different.

Approximate Visual Test



Example 13.7

□ Times on System A: {5.36, 16.57, 0.62, 1.41, 0.64, 7.26}

Times on system B: {19.12, 3.52, 3.38, 2.50, 3.60, 1.74}

$$t_{[0.95, 5]} = 2.015$$

□ The 90% confidence interval for the mean of A = 5.31 " (2.015)
= (0.24, $\sqrt{(37.92/6)}$)

□ The 90% confidence interval for the mean of B = 5.64 " (2.015)
= (0.18, $\sqrt{(44.11/6)}$)

□ Confidence intervals overlap and the mean of one falls in the confidence interval for the other.

⇒ Two systems are not different at this level of confidence.

What Confidence Level To Use?

- ❑ Need not always be 90% or 95% or 99%
- ❑ Base on the loss that you would sustain if the parameter is outside the range and the gain you would have if the parameter is inside the range.
- ❑ Low loss \Rightarrow Low confidence level is fine
E.g., lottery of 5 Million with probability 10^{-7}
- ❑ 90% confidence \Rightarrow buy nine million tickets
- ❑ 0.01% confidence level is fine.
- ❑ 50% confidence level may or may not be too low
- ❑ 99% confidence level may or may not be too high

Hypothesis Testing vs. Confidence Intervals

- ❑ Confidence interval provides more information
- ❑ Hypothesis test = yes-no decision
- ❑ Confidence interval also provides possible range
- ❑ Narrow confidence interval \Rightarrow high degree of precision
- ❑ Wide confidence interval \Rightarrow Low precision
- ❑ Example: $(-100, 100) \Rightarrow$ No difference
 $(-1, 1) \Rightarrow$ No difference
- ❑ Confidence intervals tell us not only what to say but also how loudly to say it
- ❑ CI is easier to explain to decision makers
- ❑ CI is more useful.
E.g., parameter range $(100, 200)$
vs. Probability of $(\text{parameter} = 110) = 3\%$

One Sided Confidence Intervals

- ❑ Two side intervals: 90% Confidence
 - $\Rightarrow P(\text{Difference} > \text{upper limit}) = 5\%$
 - $\Rightarrow P(\text{Difference} < \text{Lower limit}) = 5\%$
- ❑ One sided Question: Is the mean greater than 0?
 - \Rightarrow One side confidence interval
- ❑ One sided lower confidence interval for μ :
$$\left(\bar{x} - t_{[1-\alpha; n-1]} \frac{s}{\sqrt{n}}, \bar{x} \right)$$

Note t at $1-\alpha$ (not $1-\alpha/2$)
- ❑ One sided upper confidence interval for μ :
$$\left(\bar{x}, \bar{x} + t_{[1-\alpha; n-1]} \frac{s}{\sqrt{n}} \right)$$
- ❑ For large samples: Use z instead of t

Example 13.8

- Time between crashes

System	Number	Mean	Stdv
A	972	124.10	198.20
B	153	141.47	226.11

- Assume unpaired observations
- Mean difference:

$$\bar{x}_A - \bar{x}_B = 124.10 - 141.47 = -17.37$$

- Standard deviation of the difference:

$$s = \sqrt{\left(\frac{s_a^2}{n_a} + \frac{s_b^2}{n_b}\right)} = \sqrt{\frac{(198.20)^2}{972} + \frac{(226.11)^2}{153}} = 19.35$$

- Effective number of degrees of freedom:

Example 13.8 (Cont)

$$\begin{aligned}
 \nu &= \frac{\left(\frac{s_a^2}{n_a} + \frac{s_b^2}{n_b}\right)^2}{\frac{1}{n_a+1} \left(\frac{s_a^2}{n_a}\right)^2 + \frac{1}{n_b+1} \left(\frac{s_b^2}{n_b}\right)^2} - 2 \\
 &= \frac{\left(\frac{(198.20)^2}{972} + \frac{(226.11)^2}{153}\right)^2}{\frac{1}{972+1} \left(\frac{(198.20)^2}{972}\right)^2 + \frac{1}{153+1} \left(\frac{(226.11)^2}{153}\right)^2} - 2 \\
 &= 191.05
 \end{aligned}$$

- ❑ $\nu > 30 \Rightarrow$ Use z rather than t
- ❑ One sided test \Rightarrow Use $z_{0.90}=1.28$ for 90% confidence
- ❑ 90% Confidence interval:
 $(-17.37, -17.37+1.28 * 19.35)=(-17.37, 7.402)$
- ❑ CI includes zero \Rightarrow System A is not more susceptible to crashes than system B.

Confidence Intervals for Proportions

- Proportion = probabilities of various categories

E.g., $P(\text{error})=0.01$, $P(\text{No error})=0.99$

- n_1 of n observations are of type 1 \Rightarrow
Sample proportion $= p = \frac{n_1}{n}$

Confidence interval for the proportion $= p \mp z_{1-\alpha/2} \sqrt{\frac{p(1-p)}{n}}$

- Assumes Normal approximation of Binomial distribution
 \Rightarrow Valid only if $np \geq 10$.
- Need to use binomial tables if $np < 10$
Can't use t-values

CI for Proportions (Cont)

- 100(1- α)% one sided confidence interval for the proportion: ‡

$$\left(p, p + z_{1-\alpha} \sqrt{\frac{p(1-p)}{n}} \right) \text{ or } \left(p - z_{1-\alpha} \sqrt{\frac{p(1-p)}{n}}, p \right)$$

‡ Provided $np \geq 10$.

Example 13.9

- 10 out of 1000 pages printed on a laser printer are illegible.

$$\text{Sample proportion} = p = \frac{10}{1000} = 0.01$$

- $np > 10$

$$\begin{aligned}\text{Confidence interval} &= p \mp z \sqrt{\frac{p(1-p)}{n}} \\ &= 0.01 \mp z \sqrt{\frac{0.01(0.99)}{1000}} = 0.01 \mp 0.003z\end{aligned}$$

- 90% confidence interval = $0.01 \pm (1.645)(0.003)$
= (0.005, 0.015)
- 95% confidence interval = $0.01 \pm (1.960)(0.003)$
= (0.004, 0.016)

Example 13.9 (Cont)

- ❑ At 90% confidence:
0.5% to 1.5% of the pages are illegible
Chances of error = 10%
- ❑ At 95% Confidence:
0.4% to 1.6% of the pages are illegible
Chances of error = 5%

Example 13.10

- ❑ 40 Repetitions on two systems: System A superior in 26 repetitions
- ❑ Question: With 99% confidence, is system A superior?
$$p = 26/40 = 0.65$$
- ❑ Standard deviation = $\sqrt{p * (1 - p)/n} = 0.075$
- ❑ 99% confidence interval = $0.65 \pm (2.576)(0.075)$
$$= (0.46, 0.84)$$
- ❑ CI includes 0.5
 \Rightarrow we cannot say with 99% confidence that system A is superior.
- ❑ 90% confidence interval = $0.65 \pm (1.645)(0.075) = (0.53, 0.77)$
- ❑ CI does not include 0.5
 \Rightarrow Can say with 90% confidence that system A is superior.

Sample Size for Determining Mean

- Larger sample \Rightarrow Narrower confidence interval \& Higher confidence
- Question: How many observations n to get an accuracy of $\$ r\%$ and a confidence level of $100(1-\alpha)\%$?

$$\bar{x} \mp z \frac{s}{\sqrt{n}}$$

- $r\%$ Accuracy \Rightarrow
 $CI = (\bar{x}(1 - r/100), \bar{x}(1 + r/100))$

$$\bar{x} \mp z \frac{s}{\sqrt{n}} = \bar{x} \left(1 \mp \frac{r}{100}\right)$$

$$z \frac{s}{\sqrt{n}} = \bar{x} \frac{r}{100}$$

$$n = \left(\frac{100zs}{r\bar{x}} \right)^2$$

Example 13.11

- Sample mean of the response time = 20 seconds

Sample standard deviation = 5

Question: How many repetitions are needed to get the response time accurate within 1 second at 95% confidence?

- Required accuracy = 1 in 20 = 5%

Here, $\bar{x} = 20$, $s = 5$, $z = 1.960$, and $r = 5$,

$$n = \left(\frac{(100)(1.960)(5)}{(5)(20)} \right)^2 = (9.8)^2 = 96.04$$

A total of 97 observations are needed.

Sample Size for Determining Proportions

Confidence interval for the proportion = $p \mp z \sqrt{\left(\frac{p(1-p)}{n} \right)}$

To get a half-width (accuracy of) r :

$$p \mp r = p \mp z \sqrt{\left(\frac{p(1-p)}{n} \right)}$$

$$r = z \sqrt{\left(\frac{p(1-p)}{n} \right)}$$

$$n = z^2 \frac{p(1-p)}{r^2}$$

Example 13.12

- ❑ Preliminary measurement : illegible print rate of 1 in 10,000.
- ❑ Question: How many pages must be observed to get an accuracy of 1 per million at 95% confidence?
- ❑ Answer:

$$p = 1/10000 = 1E - 4, r = 1E - 6, z = 1.960$$

$$n = (1.960)^2 \left(\frac{10^{-4}(1 - 10^{-4})}{(10^{-6})^2} \right) = 384160000$$

A total of 384.16 million pages must be observed.

Example 13.13

- ❑ Algorithm A loses 0.5% of packets and algorithm B loses 0.6%.
- ❑ Question: How many packets do we need to observe to state with 95% confidence that algorithm A is better than the algorithm B?
- ❑ Answer:

$$\text{CI for algorithm A} = 0.005 \mp 1.960 \left(\frac{0.005(1 - 0.005)}{n} \right)^{1/2}$$

$$\text{CI for algorithm B} = 0.006 \mp 1.960 \left(\frac{0.006(1 - 0.006)}{n} \right)^{1/2}$$

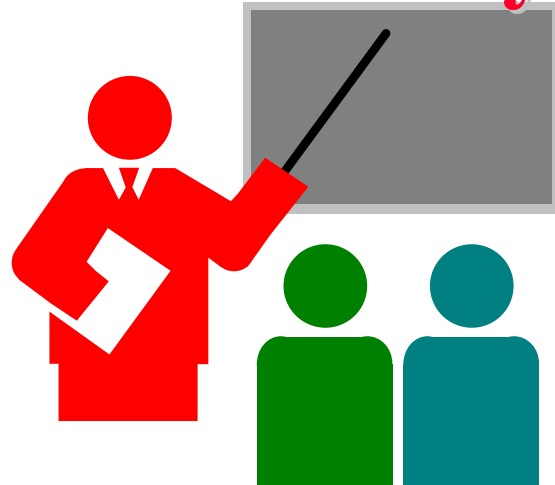
Example 13.13 (Cont)

- For non-overlapping intervals:

$$\begin{aligned} 0.005 \mp 1.960 \left(\frac{0.005(1-0.005)}{n} \right)^{1/2} \\ \leq 0.006 \mp 1.960 \left(\frac{0.006(1-0.006)}{n} \right)^{1/2} \end{aligned}$$

- $n = 84340 \Rightarrow$ We need to observe 85,000 packets.

Summary



- ❑ All statistics based on a sample are random and should be specified with a confidence interval
- ❑ If the confidence interval includes zero, the hypothesis that the population mean is zero cannot be rejected
- ❑ Paired observations \Rightarrow Test the difference for zero mean
- ❑ Unpaired observations \Rightarrow More sophisticated test
- ❑ Confidence intervals apply to proportions too.

Exercise 13.1

- Given two samples $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ from normal population $N(\mu, 1)$, what is the distribution of:
- Sample means: \bar{x}, \bar{y}
 - Difference of the means: $\bar{x} - \bar{y}$
 - Sum of the means: $\bar{x} + \bar{y}$
 - Mean of the means: $(\bar{x} + \bar{y})/2$
 - Normalized sample variances: s_x^2, s_y^2
 - Sum of sample variances: $s_x^2 + s_y^2$
 - Ratio of sample variances: s_x^2/s_y^2
 - Ratio $(\bar{x} - \mu)/s_x/\sqrt{n}$

Exercise 13.2

- ❑ Answer the following for the data of Exercise 12.1:
 - What is the 10-percentile and 90-percentile from the sample?
 - What is the mean number of disk I/Os per program?
 - What is the 90% confidence interval for the mean?
 - What fraction of programs make less than or equal to 25 I/Os and what is the 95% confidence interval for the fraction?
 - What is the one sided 90% confidence interval for the mean?

Exercise 13.3

- ❑ For the code size data of Table 11.2, find the confidence intervals for the average code sizes on various processors. Choose any two processors and answer the following:
 - At what level of significance, can you say that one is better than the other?
 - How many workloads would you need to decide the superiority at 90% confidence?

Homework

- ☐ Read chapter 13
- ☐ Submit solution to Exercise 13.2