Random Variate Generation

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Audio/Video recordings of this lecture are available at:

http://www.cse.wustl.edu/~jain/cse567-08/

1. Inverse transformation
2. Rejection
3. Composition
4. Convolution
5. Characterization

Random-Variate Generation

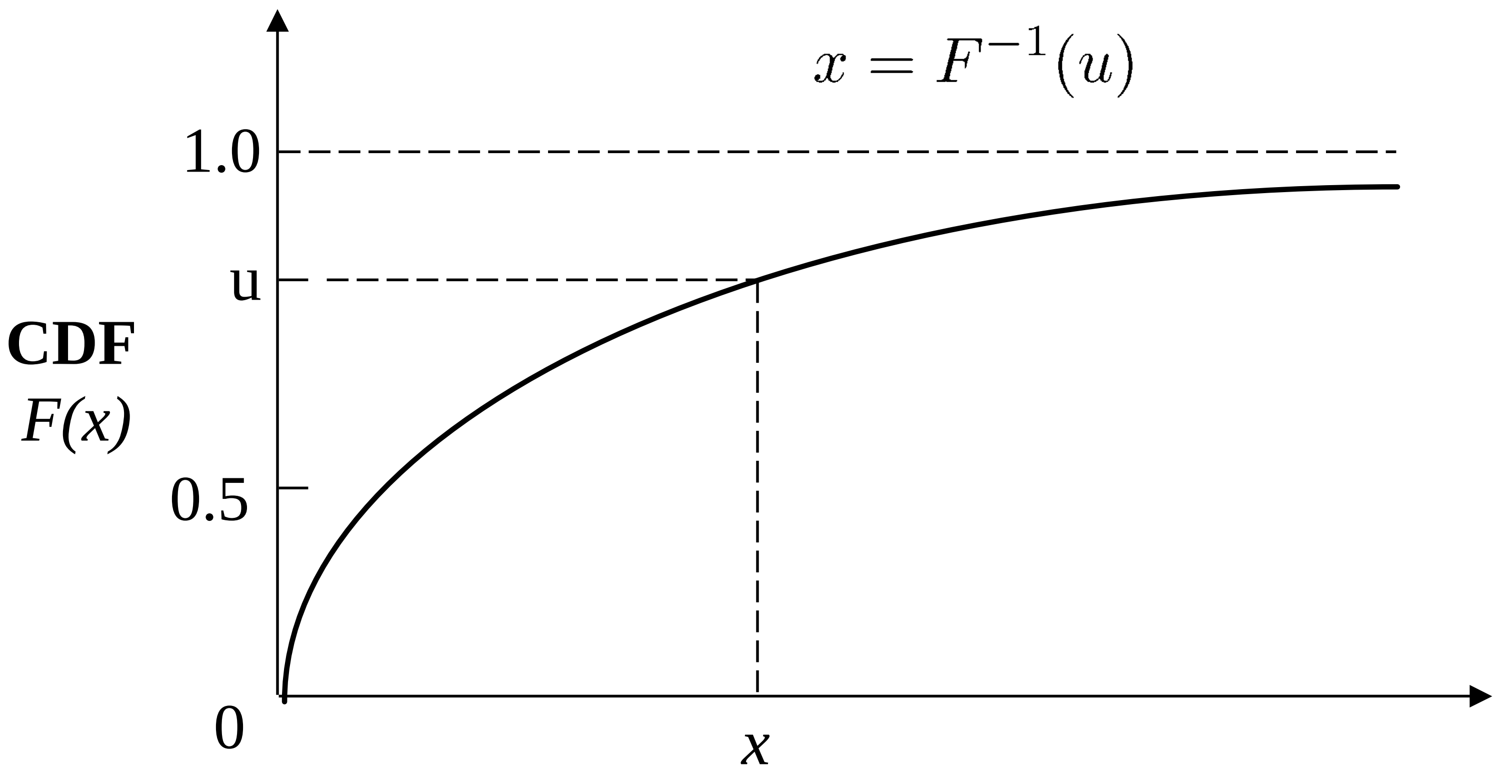
- General Techniques
- Only a few techniques may apply to a particular distribution
- Look up the distribution in Chapter 29

Inverse Transformation

- Used when F^{-1} can be determined either analytically or empirically.

$$u = F(x) \sim U(0,1)$$

$$x = F^{-1}(u)$$



Proof

Let $y = g(x)$, so that $x = g^{-1}(y)$.

$$F_Y(y) = P(Y \leq y) = P(x \leq g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

If $g(x) = F(x)$, or $y = F(x)$

$$F(y) = F(F^{-1}(y)) = y$$

And:

$$f(y) = dF/dy = 1$$

That is, y is uniformly distributed between 0 and 1.

Example 28.1

- For exponential variates:

$$\text{The pdf } f(x) = \lambda e^{-\lambda x}$$
$$\text{The CDF } F(x) = 1 - e^{-\lambda x} = u \text{ or, } x = -\frac{1}{\lambda}\ln(1-u)$$

- If u is U(0,1), 1-u is also U(0,1)
- Thus, exponential variables can be generated by:

$$x = -\frac{1}{\lambda}\ln(u)$$

Example 28.2

- The packet sizes (trimodal) probabilities:

Size	Probability
64 Bytes	0.7
128 Bytes	0.1
512 Bytes	0.2

- The CDF for this distribution is:

$$F(x) = \begin{cases} 0.0 & 0 \leq x < 64 \\ 0.7 & 64 \leq x < 128 \\ 0.8 & 128 \leq x < 512 \\ 1.0 & 512 \leq x \end{cases}$$

Example 28.2 (Cont)

- The inverse function is:

$$F^{-1}(u) = \begin{cases} 64 & 0 < u \leq 0.7 \\ 128 & 0.7 < u \leq 0.8 \\ 512 & 0.8 < u \leq 1 \end{cases}$$

$$\text{Generate } u \sim U(0,1)$$
$$u \leq 0.7 \Rightarrow Size = 64$$
$$0.7 < u \leq 0.8 \Rightarrow size = 128$$
$$0.8 < u \Rightarrow size = 512$$

- Note: CDF is *continuous from the right*
  ⇒ the value on the right of the discontinuity is used
  ⇒ The inverse function is continuous from the left
  ⇒ u=0.7 ⇒ x=64

Applications of the Inverse-Transformation Technique

Distribution	CDF $F(x)$	Inverse
Exponential	$1 - e^{-x/a}$	$-a \ln(u)$
Extreme value	$1 - e^{-e^{\frac{x-a}{b}}}$	$a + b \ln \ln u$
Geometric	$1 - (1-p)^x$	$\left\lceil \frac{\ln(u)}{\ln(1-p)} \right\rceil$
Logistic	$1 - \frac{1}{1+e^{\frac{x-\mu}{b}}}$	$\mu - b \ln(\frac{1}{u} - 1)$
Pareto	$1 - x^{-a}$	$1/u^{1/a}$
Weibull	$1 - e^{(x/a)^b}$	$a(\ln u)^{1/b}$

Rejection

- Can be used if a pdf $g(x)$ exists such that $c\,g(x)$ *majorizes* the pdf $f(x) \Rightarrow c\,g(x) \geq f(x)\ \forall\ x$
- Steps:

1. Generate x with pdf $g(x)$.
2. Generate y uniform on $[0, cg(x)]$.
3. If $y \leq f(x)$, then output x and return.
   Otherwise, repeat from step 1.
   $\Rightarrow$ Continue *rejecting* the random variates x and y until $y \geq f(x)$

- Efficiency = how closely $c\,g(x)$ envelopes $f(x)$
  Large area between $c\,g(x)$ and $f(x) \Rightarrow$ Large percentage of (x, y) generated in steps 1 and 2 are rejected
- If generation of $g(x)$ is complex, this method may not be efficient.

Example 28.2

- Beta(2,4) density function:

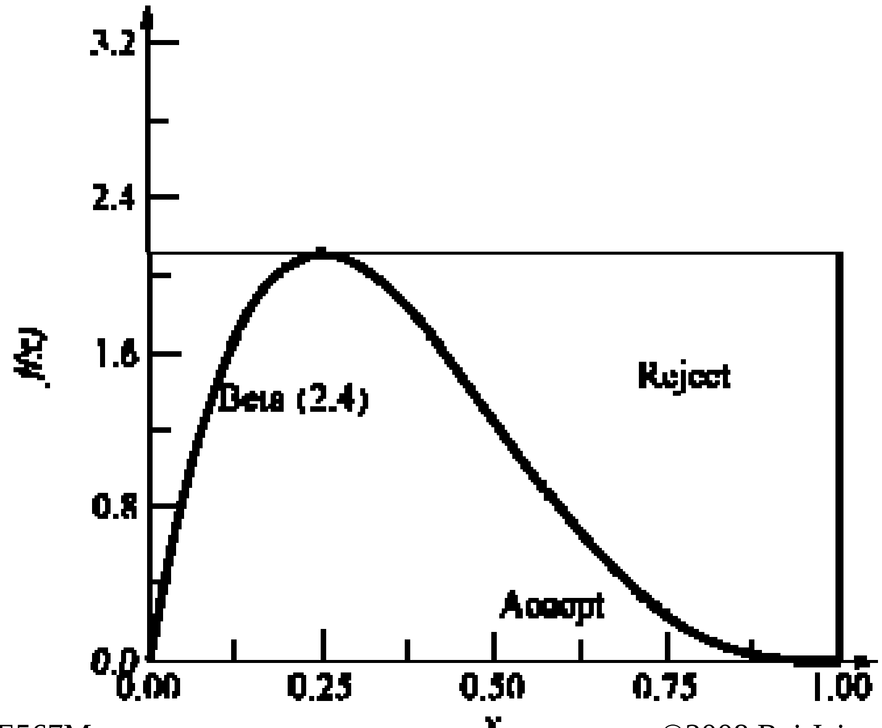
  $f(x) = 20x(1-x)^3 \quad 0 \le x \le 1$

  c=2.11 and $g(x) = 1 \quad 0 \le x \le 1$

- Bounded inside a rectangle of height 2.11

  $\Rightarrow$ Steps:

  - Generate x uniform on [0, 1].
  - Generate y uniform on [0, 2.11].
  - If $y \le 20\, x(1-x)^3$, then output *x* and return. Otherwise repeat from step 1.



Composition

- Can be used if CDF F(x) = Weighted sum of n other CDFs.

$$F(x) = \sum_{i=1}^{n} p_i F_i(x)$$

- Here, $p_i \geq 0, \sum_{i=1}^{n} p_i = 1$, and F_i's are distribution functions.
- *n* CDFs are composed together to form the desired CDF Hence, the name of the technique.
- The desired CDF is decomposed into several other CDFs $\Rightarrow$ Also called **decomposition**.
- Can also be used if the pdf *f(x)* is a weighted sum of *n* other pdfs:

$$f(x) = \sum_{i=1}^{n} p_i f_i(x)$$

Steps:

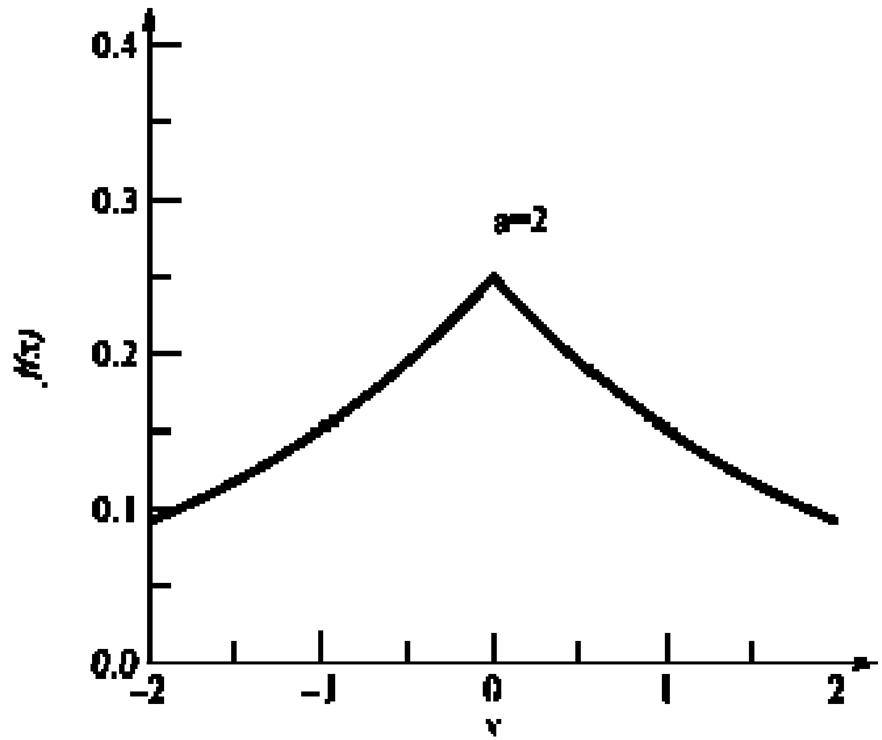
- Generate a random integer *I* such that:

$$P(I = i) = p_i$$

- This can easily be done using the inverse-transformation method.
- Generate *x* with the ith pdf $f_i(x)$ and return.

Example 28.4

- pdf: $f(x) = \frac{1}{2a} e^{-|x|/a}$
- Composition of two exponential pdf's
- Generate
  $$u_1 \sim U(0,1)$$
  $$u_2 \sim U(0,1)$$
- If u_1<0.5, return; otherwise return $x=a \ln u_2$.
- Inverse transformation better for Laplace



Convolution

- Sum of *n* variables: $x = y_1 + y_2 + \cdots + y_n$
- Generate n random variate y_i's and sum
- For sums of two variables, pdf of *x* = convolution of pdfs of y_1 and y_2. Hence the name
- Although no convolution in generation
- If pdf or CDF = Sum ⇒ Composition
- Variable x = Sum ⇒ Convolution

$$f * g(t) = \int f(\tau) g(t - \tau) d\tau$$

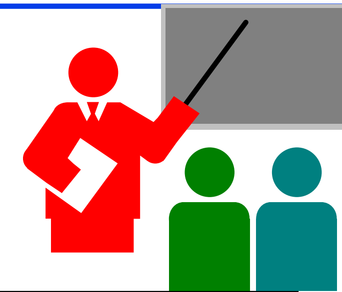
Convolution: Examples

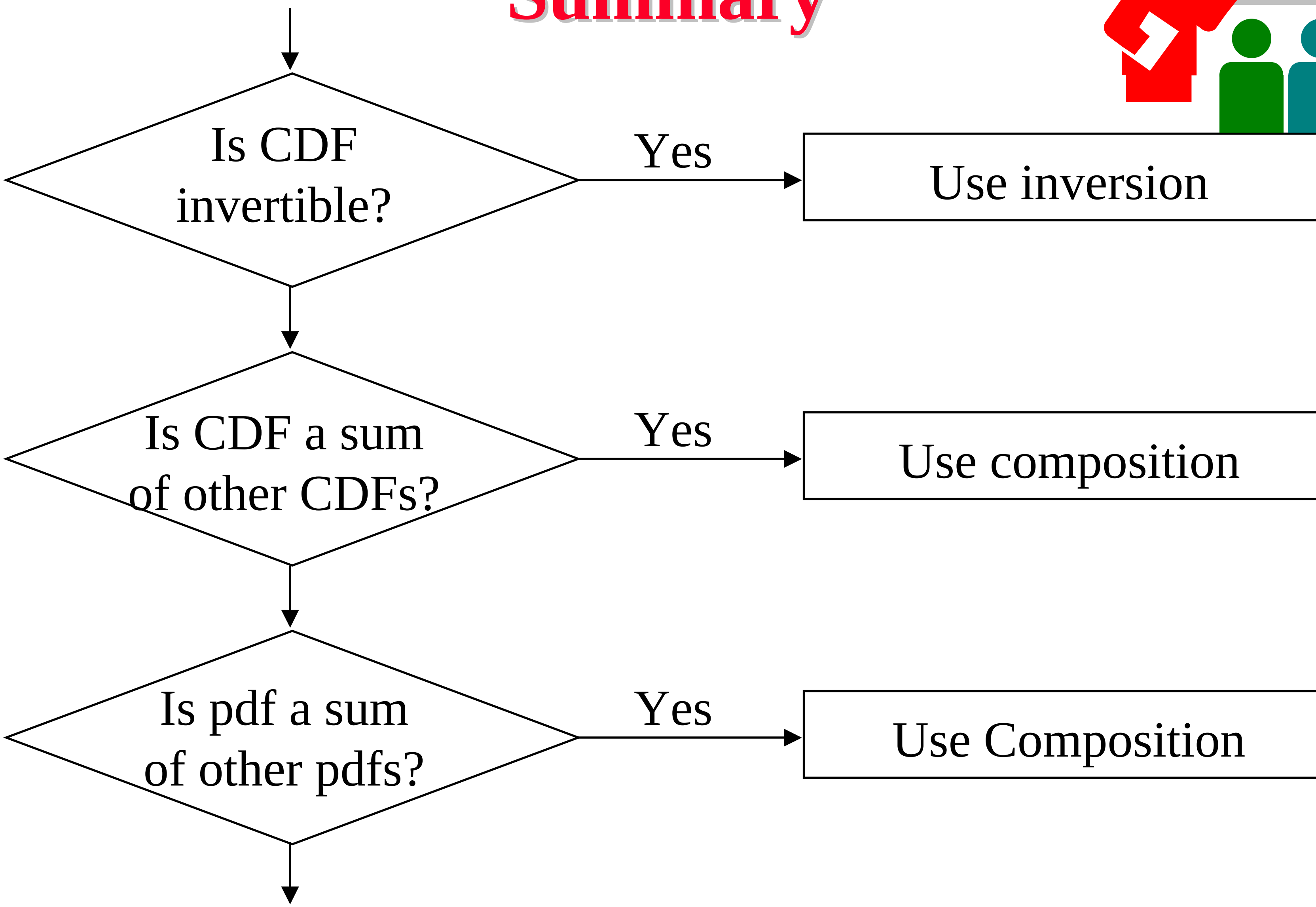
- Erlang-k = $\sum_{i=1}^{k}$ Exponential$_i$
- Binomial(n, p) = $\sum_{i=1}^{n}$ Bernoulli(p)
  $\Rightarrow$ Generated *n* U(0,1),
  return the number of RNs less than *p*
- $\chi^2(\nu) = \sum_{i=1}^{\nu} N(0,1)^2$
- $\Gamma(a, b_1)+\Gamma(a,b_2)=\Gamma(a,b_1+b_2)$
  $\Rightarrow$ Non-integer value of *b* = integer + fraction
- $\sum_{\iota=1}^{n}$ Any = Normal $\Rightarrow \sum$ U(0,1) = Normal
- $\sum_{\iota=1}^{m}$ Geometric = Pascal
- $\sum_{\iota=1}^{2}$ Uniform = Triangular

Characterization

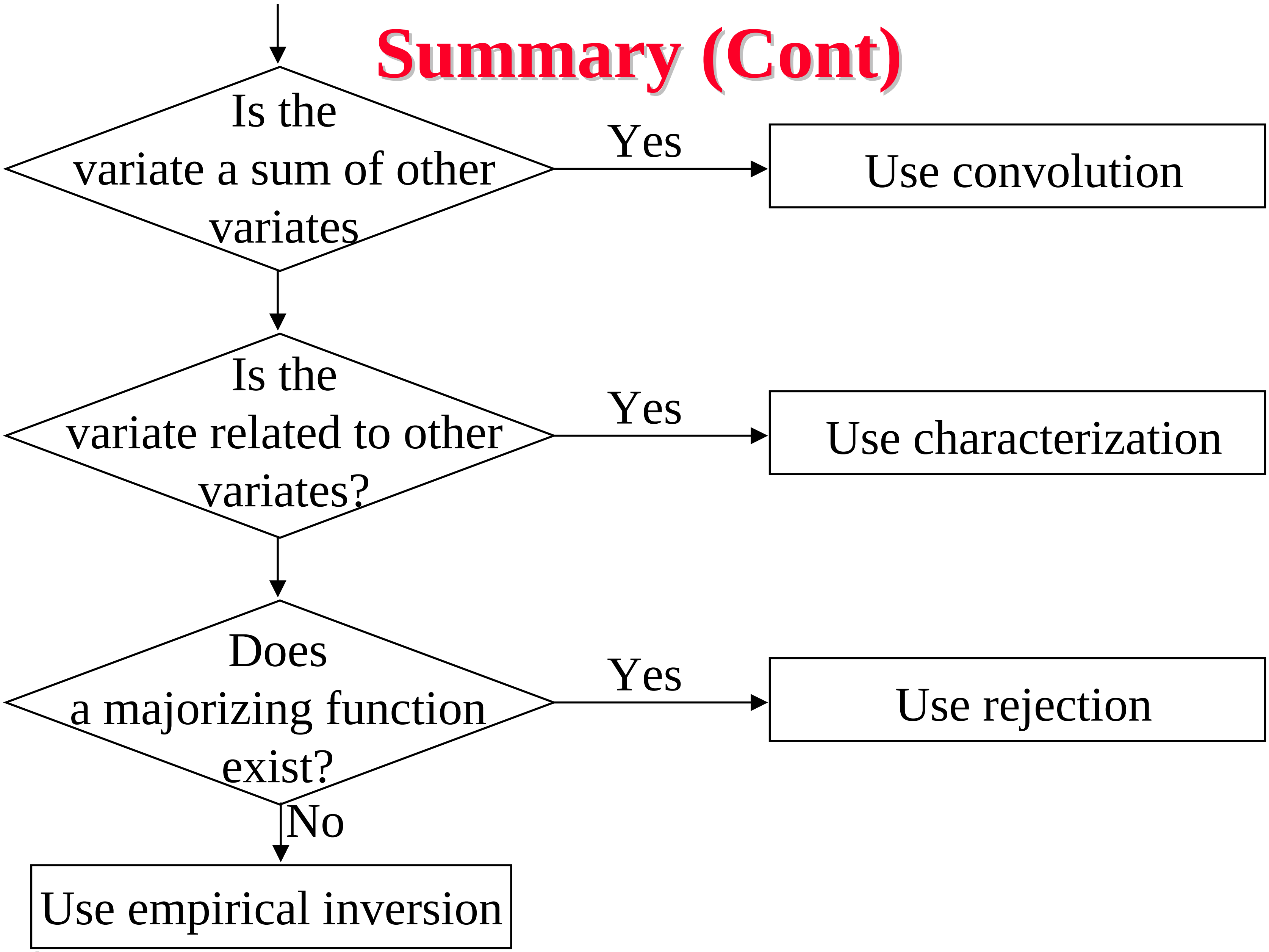
- Use special characteristics of distributions ⇒ **characterization**
- Exponential inter-arrival times ⇒ Poisson number of arrivals ⇒ Continuously generate exponential variates until their sum exceeds T and return the number of variates generated as the Poisson variate.
- The a^{th} smallest number in a sequence of $a+b+1$ U(0,1) uniform variates has a $\beta(a, b)$ distribution.
- The ratio of two unit normal variates is a Cauchy(0, 1) variate.
- A chi-square variate with even degrees of freedom $\chi^2(\nu)$ is the same as a gamma variate $\gamma(2,\nu/2)$.
- If x_1 and x_2 are two gamma variates $\gamma(a,b)$ and $\gamma(a,c)$, respectively, the ratio $x_1/(x_1+x_2)$ is a beta variate $\beta(b,c)$.
- If x is a unit normal variate, $e^{\mu+\sigma x}$ is a lognormal(μ, σ) variate.

Summary




Is CDF invertible?
Yes
Use inversion
Is CDF a sum of other CDFs?
Yes
Use composition
Is pdf a sum of other pdfs?
Yes
Use Composition

Summary (Cont)



Exercise 28.1

- A random variate has the following triangular density:

$$f(x) = \min(x, 2-x) \quad 0 \le x \le 2$$

- Develop algorithms to generate this variate using each of the following methods:

a. Inverse-transformation

b. Rejection

c. Composition

d. Convolution

Homework 28

- A random variate has the following triangular density:

$$f(x) = \frac{1}{16}\min(x, 8-x) \quad 0 \leq x \leq 8$$

- Develop algorithms to generate this variate using each of the following methods:

a. Inverse-transformation
b. Rejection
c. Composition
d. Convolution