2^{k-p} Fractional Factorial Designs

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These slides are available on-line at:

http://www.cse.wustl.edu/~jain/cse567-13/

- 2^{k-p} Fractional Factorial Designs
- Sign Table for a 2^{k-p} Design
- Confounding
- Other Fractional Factorial Designs
- Algebra of Confounding
- Design Resolution

2^{k-p} Fractional Factorial Designs

- Large number of factors
  - $\Rightarrow$ large number of experiments
  - $\Rightarrow$ full factorial design too expensive
  - $\Rightarrow$ Use a fractional factorial design
- 2^{k-p} design allows analyzing k factors with only 2^{k-p} experiments.

  2^{k-1} design requires only half as many experiments

  2^{k-2} design requires only one quarter of the experiments

Example: 2^{7-4} Design

Expt No.	A	B	C	D	E	F	G
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

- Study 7 factors with only 8 experiments!

Fractional Design Features

- Full factorial design is easy to analyze due to orthogonality of sign vectors.

  Fractional factorial designs also use orthogonal vectors. That is:

  - The sum of each column is zero.

$$\sum_i x_{ij} = 0 \quad \forall j$$

*j*th variable, *i*th experiment.

  - The sum of the products of any two columns is zero.

$$\sum_i x_{ij} x_{il} = 0 \quad \forall j \neq l$$

  - The sum of the squares of each column is 2^{7-4}, that is, 8.

$$\sum_i x_{ij}^2 = 8 \quad \forall j$$

Analysis of Frac. Factorial Designs

- **Model**:

$$
\begin{aligned}
y &= q_0 + q_A x_A + q_B x_B + q_C x_C + q_D x_D \\
&\quad + q_E x_E + q_F x_F + q_G x_G
\end{aligned}
$$

- Effects can be computed using inner products.

$$
\begin{aligned}
q_A &= \sum_i y_i x_{Ai} \\
&= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}
\end{aligned}
$$

$$
\begin{aligned}
q_B &= \sum_i y_i x_{Bi} \\
&= \frac{-y_1 - y_2 + y_3 + y_4 - y_5 - y_6 + y_7 + y_8}{8}
\end{aligned}
$$

Example 19.1

I	A	B	C	D	E	F	G	y
1	-1	-1	-1	1	1	1	-1	20
1	1	-1	-1	-1	-1	1	1	35
1	-1	1	-1	-1	1	-1	1	7
1	1	1	-1	1	-1	-1	-1	42
1	-1	-1	1	1	-1	-1	1	36
1	1	-1	1	-1	1	-1	-1	50
1	-1	1	1	-1	-1	1	-1	45
1	1	1	1	1	1	1	1	82
317	101	35	109	43	1	47	3	Total
39.62	12.62	4.37	13.62	5.37	0.125	5.87	0.37	Total/8

- Factors A through G explain 37.26%, 4.74%, 43.40%, 6.75%, 0%, 8.06%, and 0.03% of variation, respectively.

  $\Rightarrow$ Use only factors C and A for further experimentation.

Sign Table for a 2^{k-p} Design

Steps:

1. Prepare a sign table for a full factorial design with k-p factors.
2. Mark the first column I.
3. Mark the next k-p columns with the k-p factors.
4. Of the (2^{k-p}-k+p-1) columns on the right, choose p columns and mark them with the p factors which were not chosen in step 1.

Example: 2^{7-4} Design

Expt No.	A	B	C	AB	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

Example: 2^{4-1} Design

Expt No.	A	B	C	AB	AC	BC	D
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

Confounding

- **Confounding**: Only the combined influence of two or more effects can be computed.

$$
\begin{aligned}
q_A &= \sum_i y_i x_{Ai} \\
&= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}
\end{aligned}
$$

$$
\begin{aligned}
q_D &= \sum_i y_i x_{Di} \\
&= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}
\end{aligned}
$$

Confounding (Cont)

$$q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci}$$
$$= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

$$q_D = q_{ABC}$$

$$q_D + q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci}$$
$$= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

- $\Rightarrow$ Effects of D and ABC are confounded. Not a problem if q_{ABC} is negligible.

Confounding (Cont)

- Confounding representation: *D=ABC*

  Other Confoundings:

$$
\begin{aligned}
q_A &= q_{BCD} = \sum_i y_i x_{Ai} \\
&= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}
\end{aligned}
$$

$$\Rightarrow A = BCD$$

A=BCD, B=ACD, C=ABD, AB=CD, AC=BD,
BC=AD, ABC=D, and I=ABCD

- *I=ABCD* ⇒ confounding of ABCD with the mean.

Other Fractional Factorial Designs

- A fractional factorial design is not unique. 2^p different designs.

Another 2^{4-1} Experimental Design

Expt No.	A	B	C	D	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

- Confoundings: I=ABD, A=BD, B=AD, C=ABCD, D=AB, AC=BCD, BC=ACD, ABC=CD

Not as good as the previous design.

Algebra of Confounding

- Given just one confounding, it is possible to list all other confoundings.
- Rules:
  - I is treated as unity.
  - Any term with a power of 2 is erased.

$$I = ABCD$$

Multiplying both sides by A:

$$A = A^2BCD = BCD$$

Multiplying both sides by B, C, D, and AB:

Algebra of Confounding (Cont)

$$B = AB^2CD = ACD$$

$$C = ABC^2D = ABD$$

$$D = ABCD^2 = ABC$$

$$AB = A^2B^2CD = CD$$

and so on.

- Generator polynomial: $I=ABCD$

For the second design: $I=ABC$.

- In a 2^{k-p} design, 2^p effects are confounded together.

Example 19.7

- In the 2^{7-4} design:

$$D = AB, E = AC, F = BC, G = ABC$$

$$\Rightarrow I = ABD, I = ACE, I = BCF, I = ABCG$$

$$\Rightarrow I = ABD = ACE = BCF = ABCG$$

- Using products of all subsets:

$$\begin{aligned} I &= ABD = ACE = BCF = ABCG = BCDE \\ &= ACDF = CDG = ABEF = BEG \\ &= AFG = DEF = ADEG = BDFG \\ &= CEFG = ABCDEFG \end{aligned}$$

Example 19.7 (Cont)

- Other confoundings:

$$\begin{aligned} A &= BD = CE = ABCF = BCG = ABCDE \\ &= CDF = ACDG = BEF = ABEG \\ &= FG = ADEF = DEG = ABDFG \\ &= ACEFG = BCDEFG \end{aligned}$$

Design Resolution

- Order of an effect = Number of terms
  Order of $ABCD = 4$, order of $I = 0$.
- Order of a confounding = Sum of order of two terms
  E.g., AB=CDE is of order 5.
- Resolution of a Design
  = Minimum of orders of confoundings
- Notation: R_{III} = Resolution-III = 2^{k-p}_{III}
- Example 1: $I=ABCD \Rightarrow R_{IV}$ = Resolution-IV = 2^{4-1}_{IV}
  A=BCD, B=ACD, C=ABD, AB=CD, AC=BD,
  BC=AD, ABC=D, and I=ABCD

Design Resolution (Cont)

- Example 2:

  $I = ABD \Rightarrow R_{III}$ design.
- Example 3:

$$
\begin{aligned}
I &= ABD = ACE = BCF = ABCG = BCDE \\
&= ACDF = CDG = ABEF = BEG \\
&= AFG = DEF = ADEG = BDFG \\
&= CEFG = ABCDEFG
\end{aligned}
$$

- This is a resolution-III design.
- A design of higher resolution is considered a better design.

Case Study 19.1: Latex vs. troff

Factors and Levels

	Factor	-Level	+Level
A	Program	Latex	troff-me
B	Bytes	2100	25000
C	Equations	0	10
D	Floats	0	10
E	Tables	0	10
F	Footnotes	0	10

Case Study 19.1 (Cont)

- Design: 2^{6-1} with I=BCDEF

Factor		Effect	% Variation
B	Bytes	12.0	39.4%
A	Program	9.4	24.4%
C	Equations	7.5	15.6%
AC	Program × Equations	7.2	14.4%
E	Tables	3.5	3.4%
F	Footnotes	1.6	0.70%

Case Study 19.1: Conclusions

- Over 90% of the variation is due to: Bytes, Program, and Equations and a second order interaction.
- Text file size were significantly different making it's effect more than that of the programs.
- High percentage of variation explained by the ``program $\times$ Equation'' interaction
  $\Rightarrow$ Choice of the text formatting program depends upon the number of equations in the text. troff not as good for equations.

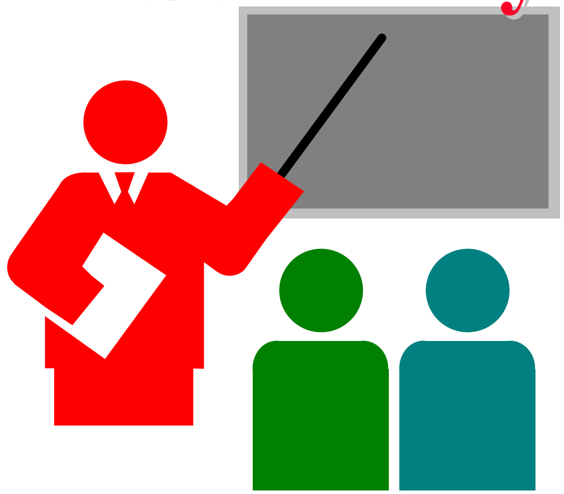
CPU Time

Program	# of Equations	
	-1(0)	1(10)
-1(Latex)	-9.7	-9.1
1(Troff)	-5.3	24.1

Case Study 19.1: Conclusions (Cont)

- Low ``Program $\times$ Bytes'' interaction $\Rightarrow$ Changing the file size affects both programs in a similar manner.
- In next phase, reduce range of file sizes. Alternately, increase the number of levels of file sizes.

Summary



- Fractional factorial designs allow a large number of variables to be analyzed with a small number of experiments
- Many effects and interactions are confounded
- The resolution of a design is the sum of the order of confounded effects
- A design with higher resolution is considered better

Exercise 19.1

Analyze the 2^{4-1} design:

		C_1		C_2	
		D_1	D_2	D_1	D_2
A_1	B_1		40	15	
	B_2		20	10	
A_2	B_1	100			30
	B_2	120			50

- Quantify all main effects.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.
- List all confoundings.
- Can you propose a better design with the same number of experiments.
- What is the resolution of the design?

Exercise 19.2

Is it possible to have a 2^{4-1}_{III} design? a 2^{4-1}_{II} design? 2^{4-1}_{IV} design? If yes, give an example.

Homework 19

- **Updated** Exercise 19.1
  Analyze the 2^{4-1} design:

		C_1		C_2	
		D_1	D_2	D_1	D_2
A_1	B_1		30	15	
	B_2		20	10	
A_2	B_1	100			30
	B_2	110			50

- Quantify all main effects.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.
- List all confoundings.
- Can you propose a better design with the same number of experiments.
- What is the resolution of the design?