Two Factor Full Factorial Design with Replications

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These slides are available on-line at:

http://www.cse.wustl.edu/~jain/cse567-13/

- Model
- Computation of Effects
- Estimating Experimental Errors
- Allocation of Variation
- ANOVA Table and F-Test
- Confidence Intervals For Effects

Model

- Replications allow separating out the interactions from experimental errors.
- Model: With *r* replications

$$y_{ijk} = \mu + \alpha_j + \beta_i + \gamma_{ij} + e_{ijk}$$

$$\begin{array}{lcl} y_{ijk} & = & \text{Response in the kth replication} \\ & & \text{with factor A at level j and factor B at level i} \\ \mu & = & \text{mean response} \\ \alpha_j & = & \text{Effect of factor A at level j} \\ \beta_i & = & \text{Effect of Factor B at level i} \\ \gamma_{ij} & = & \text{Effect of interaction between factors A and B} \\ e_{ijk} & = & \text{Experimental error} \end{array}$$

Model (Cont)

- The effects are computed so that their sum is zero:

$$\sum_{j=1}^{a} \alpha_j = 0; \sum_{i=1}^{b} \beta_i = 0;$$

- The interactions are computed so that their row as well as column sums are zero:

$$\sum_{j=1}^{a} \gamma_{1j} = \sum_{j=1}^{a} \gamma_{2j} = \cdots = \sum_{j=1}^{a} \gamma_{bj} = 0$$

$$\sum_{i=1}^{b} \gamma_{i1} = \sum_{i=1}^{b} \gamma_{i2} = \cdots = \sum_{i=1}^{b} \gamma_{ia} = 0$$

- The errors in each experiment add up to zero:

$$\sum_{k=1}^{r} e_{ijk} = 0 \quad \forall i, j$$

Computation of Effects

$$y_{ijk} = \mu + \alpha_j + \beta_i + \gamma_{ij} + e_{ijk}$$

- Averaging the observations in each cell:

$$\bar{y}_{ij.} = \mu + \alpha_j + \beta_i + \gamma_{ij}$$

- Similarly,

$$\bar{y}_{i..} = \mu + \beta_i$$

$$\bar{y}_{.j.} = \mu + \alpha_j$$

$$\bar{y}_{...} = \mu$$

$\Rightarrow$

$$\mu = \bar{y}_{...}$$

$$\alpha_j = \bar{y}_{.j.} - \bar{y}_{...}$$

$$\beta_i = \bar{y}_{i..} - \bar{y}_{...}$$

$$\gamma_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

$\Rightarrow$ Use cell means to compute row and column effects.

Example 22.1: Code Size

	Processors			
Workloads	W	X	Y	Z
I	7006	12042	29061	9903
	6593	11794	27045	9206
	7302	13074	30057	10035
J	3207	5123	8960	4153
	2883	5632	8064	4257
	3523	4608	9677	4065
K	4707	9407	19740	7089
	4935	8933	19345	6982
	4465	9964	21122	6678
L	5107	5613	22340	5356
	5508	5947	23102	5734
	4743	5161	21446	4965
W	6807	12243	28560	9803
	6392	11995	26846	9306
	7208	12974	30559	10233

Example 22.1: Log Transformation

Workloads	Processors W	X	Y	Z
I	3.8455	4.0807	4.4633	3.9958
	3.8191	4.0717	4.4321	3.9641
	3.8634	4.1164	4.4779	4.0015
J	3.5061	3.7095	3.9523	3.6184
	3.4598	3.7507	3.9066	3.6291
	3.5469	3.6635	3.9857	3.6091
K	3.6727	3.9735	4.2953	3.8506
	3.6933	3.9510	4.2866	3.8440
	3.6498	3.9984	4.3247	3.8246
L	3.7082	3.7492	4.3491	3.7288
	3.7410	3.7743	4.3636	3.7585
	3.6761	3.7127	4.3313	3.6959
M	3.8330	4.0879	4.4558	3.9914
	3.8056	4.0790	4.4289	3.9688
	3.8578	4.1131	4.4851	4.0100

Example 22.1: Computation of Effects

Workloads	Processors W	X	Y	Z	Row Sum	Row Mean	Row Effect
I	3.8427	4.0896	4.4578	3.9871	16.3772	4.0943	0.1520
J	3.5043	3.7079	3.9482	3.6188	14.7792	3.6948	-0.2475
K	3.6720	3.9743	4.3022	3.8397	15.7882	3.9470	0.0047
L	3.7084	3.7454	4.3480	3.7277	15.5295	3.8824	-0.0599
M	3.8321	4.0933	4.4566	3.9900	16.3720	4.0930	0.1507
Col Sum	18.5594	19.6105	21.5128	19.1635	78.8463		
Col Mean	3.7119	3.9221	4.3026	3.8327		3.9423	
Col effect	-0.2304	-0.0202	0.3603	-0.1096			

- An average workload on an average processor requires a code size of $10^{3.94}$ (8710 instructions).
- Processor W requires $10^{0.23}$ (=1.69) less code than avg processor.
- Processor X requires $10^{0.02}$ (=1.05) less than an average processor and so on.
- The ratio of code sizes of an average workload on processor W and X is $10^{0.21}$ (= 1.62).

Example 22.1: Interactions

Workloads	W	X	Y	Z
I	-0.0212	0.0155	0.0032	0.0024
J	0.0399	0.0333	-0.1069	0.0337
K	-0.0447	0.0475	-0.0051	0.0023
L	0.0564	-0.1168	0.1054	-0.0450
M	-0.0305	0.0205	0.0033	0.0066

- Check: The row as well column sums of interactions are zero.
- Interpretation: Workload I on processor W requires 0.02 less log code size than an average workload on processor W or equivalently 0.02 less log code size than I on an average processor.

Computation of Errors

- Estimated Response:

$$\hat{y}_{ij} = \mu + \alpha_j + \beta_i + \gamma_{ij} = \bar{y}_{ij.}$$

- Error in the *k*th replication:

$$e_{ijk} = y_{ijk} - \bar{y}_{ij.}$$

- **Example 22.2**: Cell mean for (1,1) = 3.8427
  Errors in the observations in this cell are:
  3.8455-3.8427 = 0.0028
  3.8191-3.8427 = -0.0236, and
  3.8634-3.8427 = 0.0208
  Check: Sum of the three errors is zero.

Allocation of Variation

$$
\begin{array}{ccccccccccc}
\sum_{ijk} y_{ijk}^2 & = & abr\mu^2 & + & br\sum_j \alpha_j^2 & + & ar\sum_i \beta_i^2 & + & r\sum_{ij} \gamma_{ij}^2 & + & \sum_{ijk} e_{ijk}^2 \\
\text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSAB} & + & \text{SSE}
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
\text{SST} & = & \text{SSY} & - & \text{SS0} & = & \text{SSA} & + & \text{SSB} & + & \text{SSAB} & + & \text{SSE} \\
4.44 & = & 936.95 & - & 932.51 & = & 2.93 & + & 1.33 & + & 0.15 & + & 0.03 \\
100\% & = & & & & = & 65.96\% & + & 29.9\% & + & 3.48\% & + & 0.66\%
\end{array}
$$

- Interactions explain less than 5% of variation
  $\Rightarrow$ may be ignored.

Analysis of Variance

- Degrees of freedoms:

$$
\begin{array}{ccccccccccc}
\mathrm{SSY} & = & \mathrm{SS0} & + & \mathrm{SSA} & + & \mathrm{SSB} & + & \mathrm{SSAB} & + & \mathrm{SSE} \\
abr & = & 1 & + & (a-1) & + & (b-1) & + & (a-1)(b-1) & + & ab(r-1)
\end{array}
$$

$$
\begin{aligned}
\frac{\mathrm{MSA}}{\mathrm{MSE}} &\sim F[a-1, ab(r-1)] \\
\frac{\mathrm{MSB}}{\mathrm{MSE}} &\sim F[b-1, ab(r-1)] \\
\frac{\mathrm{MSAB}}{\mathrm{MSE}} &\sim F[(a-1)(b-1), ab(r-1)]
\end{aligned}
$$

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ANOVA for Two Factors w Replications

Component	Sum of Squares	%Variation	DF	Mean Square	F-Comp.	F-Table
y	$\text{SSY} = \sum y_{ijk}^2$		abr			
$\bar{y}_{\ldots}$	$\text{SS0} = abr\mu^2$		1			
$y - \bar{y}_{\ldots}$	$\text{SST} = \text{SSY} - \text{SS0}$	100	$abr - 1$			
A	$\text{SSA} = br\Sigma\alpha_j^2$	$100\left(\frac{\text{SSA}}{\text{SST}}\right)$	$a-1$	$\text{MSA} = \frac{\text{SSA}}{a-1}$	$\frac{\text{MSA}}{\text{MSE}}$	$F_{[1-\alpha;a-1,ab(r-1)]}$
B	$\text{SSB} = ar\Sigma\beta_i^2$	$100\left(\frac{\text{SSB}}{\text{SST}}\right)$	$b-1$	$\text{MSB} = \frac{\text{SSB}}{b-1}$	$\frac{\text{MSB}}{\text{MSE}}$	$F_{[1-\alpha;b-1,ab(r-1)]}$
AB	$\text{SSAB} = r\Sigma\gamma_{ij}^2$	$100\left(\frac{\text{SSAB}}{\text{SST}}\right)$	$(a-1)(b-1)$	$\text{MSAB} = \frac{\text{SSAB}}{(a-1)(b-1)}$	$\frac{\text{MSA}}{\text{MSE}}$	$F_{[1-\alpha,(a-1)(b-1),ab(r-1)]}$
e	$\text{SSE} = \text{SST} - (\text{SSA} + \text{SSB} + \text{SSAB})$	$100\left(\frac{\text{SSE}}{\text{SST}}\right)$	$ab(r-1)$	$\text{MSE} = \frac{\text{SSE}}{ab(r-1)}$		

Example 22.4: Code Size Study

Component	Sum of Squares	%Variation	DF	Mean Square	F-Comp.	F-Table
y	936.95					
$\bar{y}_{...}$	932.51					
$y - \bar{y}_{...}$	4.44	100.00%	59			
Processors	2.93	65.96%	3	0.9765	1340.01	2.23
Workloads	1.33	29.90%	4	0.3320	455.65	2.09
Interactions	0.15	3.48%	12	0.0129	17.70	1.71
Errors	0.03	0.66%	40	0.0007		

$$s_e = \sqrt{\text{MSE}} = \sqrt{0.0008} = 0.03$$

- All three effects are statistically significant at a significance level of 0.10.

Confidence Intervals For Effects

Parameter Estimation

Parameter	Estimate	Variance
μ	$\bar{y}_{...}$	s_e^2/abr
α_j	$\bar{y}_{i..}$-$\bar{y}_{...}$	$s_e^2(a-1)/abr$
β_i	$\bar{y}_{.j.}$-$\bar{y}_{...}$	$s_e^2(b-1)/abr$
γ_{ij}	$\bar{y}_{ij.}$-$\bar{y}_{i..}$-$\bar{y}_{.j.}$+$\bar{y}_{...}$	$s_e^2(a-1)(b-1)/abr$
$\Sigma\ h_j\ \alpha_j,\ \Sigma\ h_j=0$	$\Sigma\ h_j\ \bar{y}_{.j.}$	$\Sigma\ h_j^2 s_e^2/br$
$\Sigma\ h_i\ \beta_i,\ \Sigma\ h_i=0$	$\Sigma\ h_i\ \bar{y}_{i..}$	$\Sigma\ h_i^2 s_e^2/ar$
s_e^2	$\Sigma\ e_{ijk}^2/\{ab(r-1)\}$	

Degrees of freedom for errors = ab(r-1)

- Use *t* values at *ab(r-1)* degrees of freedom for confidence intervals

Example 22.5: Code Size Study

- From ANOVA table: s_e=0.03. The standard deviation of processor effects:

$$s_{\alpha_j} = s_e\sqrt{\frac{a-1}{abr}} = 0.03\sqrt{\frac{4-1}{4\times 5\times 3}} = 0.0060$$

- The error degrees of freedom:

  $ab(r-1) = 40 \Rightarrow$ use Normal tables

  For 90% confidence, $z_{0.95} = 1.645$

  90% confidence interval for the effect of processor W is:

  $$\begin{aligned}\alpha_1 \mp t\, s_{\alpha_1} &= -0.2304 \mp 1.645 \times 0.0060\\ &= -0.2304 \mp 0.00987\\ &= (-0.2406, -0.2203)\end{aligned}$$

  The effect is significant.

Example 22.5: Conf. Intervals (Cont)

Parameter	Mean Effect	Std. Dev.	Confidence Interval
μ	3.9423	0.0035	(3.9364, 3.9482)
Processors			
W	-0.2304	0.0060	(-0.2406, -0.2203)
X	-0.0202	0.0060	(-0.0304, -0.0100)
Y	0.3603	0.0060	(0.3501, 0.3704)
Z	-0.1096	0.0060	(-0.1198, -0.0995)
Workloads			
I	0.1520	0.0070	(0.1402, 0.1637)
J	-0.2475	0.0070	(-0.2592, -0.2358)
K	0.0047	0.0070	(-0.0070, 0.0165)†
L	-0.0599	0.0070	(-0.0717, -0.0482)
M	0.1507	0.0070	(0.1390, 0.1624)

† $\Rightarrow$ Not significant

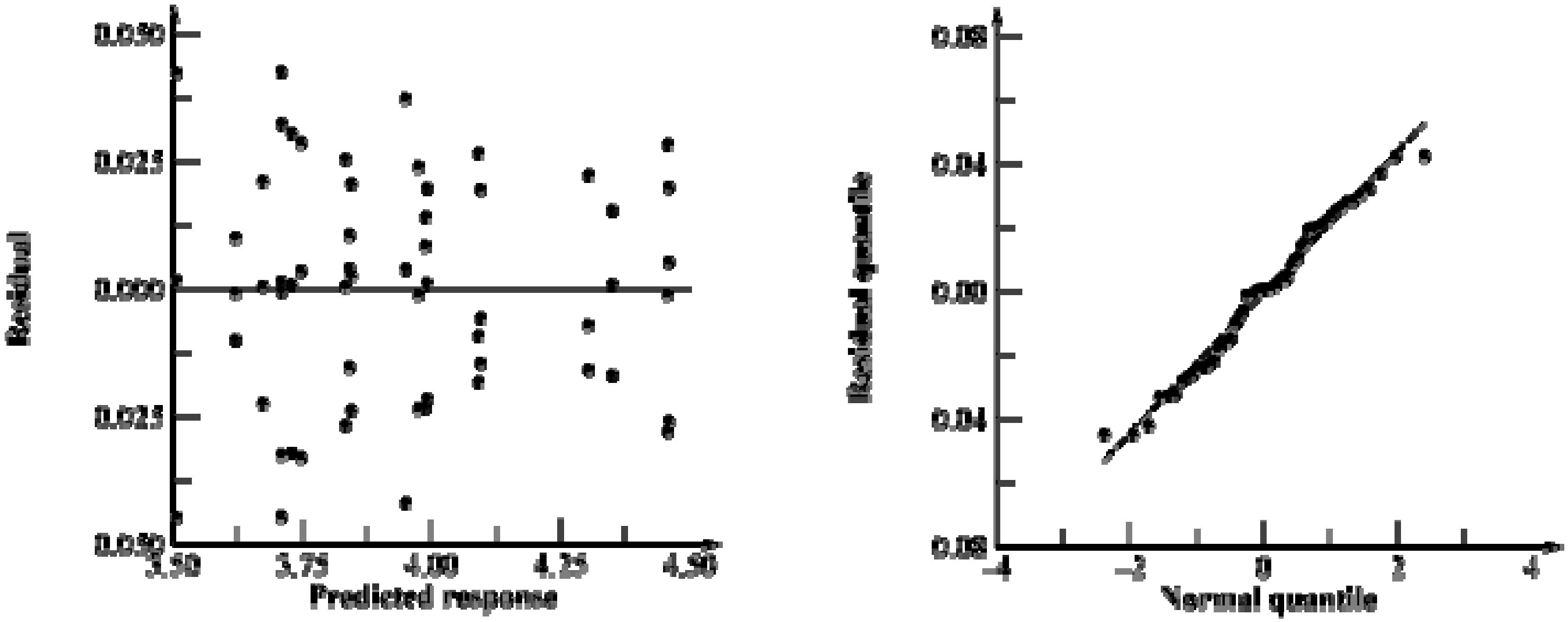
- The intervals are very narrow.

Example 22.5: CI for Interactions

Workloads	W	X	Y	Z
I	(-0.0415, -0.0009)	(-0.0048, 0.0358)†	(-0.0171, 0.0236)†	(-0.0179, 0.0228)†
J	(0.0196, 0.0602)	(0.0130, 0.0536)	(-0.1272, -0.0865)	(0.0133, 0.0540)
K	(-0.0650, -0.0243)	(0.0271, 0.0678)	(-0.0254, 0.0152)†	(-0.0180, 0.0226)†
L	(0.0361, 0.0768)	(-0.1371, -0.0964)	(0.0850, 0.1257)	(-0.0654, -0.0247)
M	(-0.0508, -0.0101)	(0.0002, 0.0408)	(-0.0170, 0.0236)†	(-0.0137, 0.0270)†

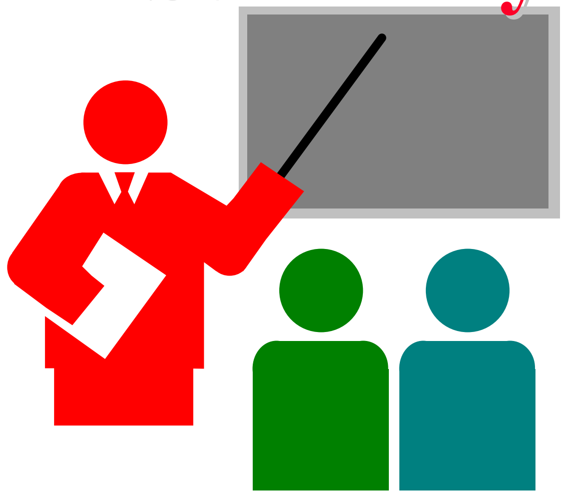
† $\Rightarrow$ Not significant

Example 22.5: Visual Tests



- No visible trend.
- Approximately linear ⇒ normality is valid.

Summary



- Replications allow interactions to be estimated

$$y_{ijk} = \mu + \alpha_j + \beta_i + \gamma_{ij} + e_{ijk}$$

$$\sum_{j=1}^{a} \alpha_j = 0; \quad \sum_{i=1}^{b} \beta_i = 0; \quad \sum_{j=1}^{a} \gamma_{ij} = 0 \ \forall i;$$

$$\sum_{i=1}^{b} \gamma_{ij} = 0 \ \forall j; \sum_{k=1}^{r} e_{ijk} = 0 \ \forall i, j$$

- SSE has ab(r-1) degrees of freedom
- Need to conduct F-tests for MSA/MSE, MSB/MSE, MSAB/MSE

Exercise 22.1

Measured CPU times for three processors A1, A2, and A3, on five workloads B1, B2, through B5 are shown in the table. Three replications of each experiment are shown. Analyze the data and answer the following:

- Are the processors different from each other at 90% level of confidence?
- What percent of variation is explained by the processor-workload interaction?
- Which effects in the model are not significant at 90% confidence.

	A		
B	A1	A2	A3
B1	3200	5120	8960
	3150	5100	8900
	3250	5140	8840
B2	4700	9400	19740
	4740	9300	19790
	4660	9500	19690
B3	3200	4160	7360
	3220	4100	7300
	3180	4220	7420
B4	5100	5610	22340
	5200	5575	22440
	5000	5645	22540
B5	6800	12240	28560
	6765	12290	28360
	6835	12190	28760

Homework 22

- Submit answer to Exercise 22.1. Show all numerical values.