

# $2^k_r$ Factorial Designs

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<http://www.cse.wustl.edu/~jain/cse567-15/>



- ❑ Computation of Effects
- ❑ Estimation of Experimental Errors
- ❑ Allocation of Variation
- ❑ Confidence Intervals for Effects
- ❑ Confidence Intervals for Predicted Responses
- ❑ Visual Tests for Verifying the assumptions
- ❑ Multiplicative Models

# $2^k r$ Factorial Designs

- $r$  replications of  $2^k$  Experiments  
⇒  $2^k r$  observations.  
⇒ Allows estimation of experimental errors.

- Model:

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e$$

- $e$  = Experimental error

# Computation of Effects

Simply use means of r measurements

I	A	B	A B	y	Mean $\bar{y}$
1	-1	-1	1	(15, 18, 12)	15
1	1	-1	-1	(45, 48, 51)	48
1	-1	1	-1	(25, 28, 19)	24
1	1	1	1	(75, 75, 81)	77
164	86	38	20		total
41	21.5	9.5	5		total/4

□ Effects:  $q_0 = 41$ ,  $q_A = 21.5$ ,  $q_B = 9.5$ ,  $q_{AB} = 5$ .

# Estimation of Experimental Errors

□ Estimated Response:

$$\hat{y}_i = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi}$$

Experimental Error = Measured - Estimated

$$\begin{aligned} e_{ij} &= y_{ij} - \hat{y}_i \\ &= y_{ij} - q_0 - q_A x_{Ai} - q_B x_{Bi} - q_{AB} x_{Ai} x_{Bi} \\ \sum_{i,j} e_{ij} &= 0 \end{aligned}$$

- Sum of Squared Errors: 
$$SSE = \sum_{i=1}^{2^2} \sum_{j=1}^r e_{ij}^2$$

# Experimental Errors: Example

- Estimated Response:

$$\hat{y}_1 = q_0 - q_A - q_B + q_{AB} = 41 - 21.5 - 9.5 + 5 = 15$$

- Experimental errors:

$$e_{11} = y_{11} - \hat{y}_1 = 15 - 15 = 0$$

i	Effect				Estimated Response	Measured Responses			Errors		
	I	A	B	A B		$\hat{y}_i$	$y_{i1}$	$y_{i2}$	$y_{i3}$	$e_{i1}$	$e_{i2}$
	41	21.5	9.5	5							
1	1	-1	-1	1	15	15	18	12	0	3	-3
2	1	1	-1	-1	48	45	48	51	-3	0	3
3	1	-1	1	-1	24	25	28	19	1	4	-5
4	1	1	1	1	77	75	75	81	-2	-2	4

# Allocation of Variation

- Total variation or total sum of squares:

$$SST = \sum_{i,j} (y_{ij} - \bar{y}_{..})^2$$

$$y_{ij} = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi} + e_{ij}$$

$$\begin{aligned} \sum_{i,j} (y_{ij} - \bar{y}_{..})^2 &= 2^2 r q_A^2 + 2^2 r q_B^2 + 2^2 r q_{AB}^2 + \sum_{i,j} e_{ij}^2 \\ SST &= SSA + SSB + SSAB + SSE \end{aligned}$$

## Example 18.3: Memory-Cache Study

$$\begin{aligned} \text{SSY} &= 15^2 + 18^2 + 12^2 + 45^2 + \dots + 75^2 + 75^2 + 81^2 \\ &= 27204 \end{aligned}$$

$$\text{SS0} = 2^2 r q_0^2 = 12 \times 41^2 = 20172$$

$$\text{SSA} = 2^2 r q_A^2 = 12 \times (21.5)^2 = 5547$$

$$\text{SSB} = 2^2 r q_B^2 = 12 \times (9.5)^2 = 1083$$

$$\text{SSAB} = 2^2 r q_{AB}^2 = 12 \times 5^2 = 300$$

$$\begin{aligned} \text{SSE} &= 27204 - 2^2 \times 3(41^2 + 21.5^2 + 9.5^2 + 5^2) \\ &= 102 \end{aligned}$$

$$\begin{aligned} \text{SST} &= \text{SSY} - \text{SS0} \\ &= 27204 - 20172 = 7032 \end{aligned}$$



## Example 18.3 (Cont)

$$\begin{aligned} &SSA + SSB + SSAB + SSE \\ &= 5547 + 1083 + 300 + 102 \\ &= 7032 = SST \end{aligned}$$

Factor A explains  $5547/7032$  or 78.88%

Factor B explains 15.40%

Interaction AB explains 4.27%

1.45% is unexplained and is attributed to errors.

# Confidence Intervals For Effects

- Effects are random variables.
- Errors  $\sim N(0, \sigma_e) \Rightarrow y \sim N(\bar{y}_{\cdot}, \sigma_e)$

$$q_0 = \frac{1}{2^{2r}} \sum_{i,j} y_{ij}$$

- $q_0$  = Linear combination of normal variates  
 $\Rightarrow q_0$  is normal with variance  $\sigma_e^2/(2^{2r})$

Variance of errors:

$$s_e^2 = \frac{1}{2^{2(r-1)}} \sum_{ij} e_{ij}^2 = \frac{\text{SSE}}{2^{2(r-1)}} \triangleq \text{MSE}$$

- Denominator =  $2^{2(r-1)}$  = # of independent terms in SSE  
 $\Rightarrow$  SSE has  $2^{2(r-1)}$  degrees of freedom.  
Estimated variance of  $q_0$ :  $s_{q_0}^2 = s_e^2/(2^{2r})$

# Conf. Intervals For Effects (Cont)

- Similarly,

$$s_{q_A} = s_{q_B} = s_{q_{AB}} = \frac{s_e}{\sqrt{2^2 r}}$$

- Confidence intervals (CI) for the effects:

$$q_i \mp t_{[1-\alpha/2; 2^2(r-1)]} s_{q_i}$$

- CI does not include a zero  $\Rightarrow$  significant

## Example 18.4

- For Memory-cache study: Standard deviation of errors:

$$s_e = \sqrt{\frac{\text{SSE}}{2^2(r-1)}} = \sqrt{\frac{102}{8}} = \sqrt{12.75} = 3.57$$

- Standard deviation of effects:

$$s_{q_i} = s_e / \sqrt{(2^2 r)} = 3.57 / \sqrt{12} = 1.03$$

- For 90% Confidence:  $t_{[0.95,8]} = 1.86$

- Confidence intervals:  $q_i \mp (1.86)(1.03) = q_i \mp 1.92$

$$q_0 = (39.08, 42.91)$$

$$q_A = (19.58, 23.41)$$

$$q_B = (7.58, 11.41)$$

$$q_{AB} = (3.08, 6.91)$$

- No zero crossing  $\Rightarrow$  All effects are significant.

# Confidence Intervals for Contrasts

- Contrast: Linear combination with  $\sum$  coefficients = 0

$$\sum h_i q_i \text{ with } \sum h_i = 0$$

For example,  $q_A - q_B$  or  $q_A + q_B - 2q_{AB}$

- Mean of  $\sum h_i q_i = \sum h_i E[q_i]$

- Variance of  $\sum h_i q_i$   $s_{\sum h_i q_i}^2 = \frac{s_e^2 \sum h_i^2}{2^2 r}$

- For  $100(1-\alpha)\%$  confidence interval, use  $t_{[1-\alpha/2; 2^2(r-1)]}$ .

## Example 18.5

Memory-cache study

$$u = q_A + q_B - 2q_{AB}$$

Coefficients = 0, 1, 1, and -2  $\Rightarrow$  Contrast

$$\text{Mean } \bar{u} = 21.5 + 9.5 - 2 \times 5 = 21$$

$$\text{Variance } s_u^2 = \frac{s_e^2 \times 6}{2^2 \times 3} = 6.375$$

$$\text{Standard deviation } s_u = \sqrt{6.375} = 2.52$$

$$t_{[0.95;8]} = 1.86$$

90% Confidence interval for u:

$$\bar{u} \mp ts_u = 21 \mp 1.86 \times 2.52 = (16.31, 25.69)$$

# Conf. Interval For Predictions

- Mean response  $\hat{y}$ :

$$\hat{y} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

- The standard deviation of the mean of m responses:

$$s_{\hat{y}_m} = s_e \left( \frac{1}{n_{\text{eff}}} + \frac{1}{m} \right)^{1/2}$$

$$\begin{aligned} n_{\text{eff}} &= \text{Effective deg of freedom} \\ &= \frac{\text{Total number of runs}}{1 + \text{Sum of DFs of params used in } \hat{y}} \\ &= \frac{2^2 r}{5} \end{aligned}$$

# Conf. Interval for Predictions (Cont)

100(1- $\alpha$ )% confidence interval:

$$\hat{y} \mp t_{[1-\alpha/2; 2^2(r-1)]} s_{\hat{y}_m}$$

- A single run ( $m=1$ ):  $s_{\hat{y}_1} = s_e \left( \frac{5}{2^2 r} + 1 \right)^{1/2}$
- Population mean ( $m=\infty$ ):  $s_{\hat{y}} = s_e \left( \frac{5}{2^2 r} \right)^{1/2}$



## Example 18.6: Memory-cache Study

- For  $x_A = -1$  and  $x_B = -1$ :
- A single confirmation experiment:

$$\begin{aligned}\hat{y}_1 &= q_0 - q_A - q_B + q_{AB} \\ &= 41 - 21.5 - 9.5 + 5 = 15\end{aligned}$$

- Standard deviation of the prediction:

$$s_{\hat{y}_1} = s_e \left( \frac{5}{2^2 r} + 1 \right)^{1/2} = 3.57 \sqrt{\frac{5}{12} + 1} = 4.25$$

- Using  $t_{[0.95;8]} = 1.86$ , the 90% confidence interval is:

$$15 \mp 1.86 \times 4.25 = (7.09, 22.91)$$

## Example 18.6 (Cont)

- Mean response for 5 experiments in future:

$$\begin{aligned} s_{\hat{y}_1} &= s_e \left( \frac{5}{2^2 r} + \frac{1}{m} \right)^{1/2} \\ &= 3.57 \sqrt{\frac{5}{12} + \frac{1}{5}} = 2.80 \end{aligned}$$

- The 90% confidence interval is:

$$15 \mp 1.86 \times 2.80 = (9.79, 20.21)$$

## Example 18.6 (Cont)

- Mean response for a large number of experiments in future:

$$s_{\hat{y}_1} = s_e \left( \frac{5}{2^2 r} \right)^{1/2} = 3.57 \sqrt{\frac{5}{12}} = 2.30$$

- The 90% confidence interval is:

$$15 \mp 1.86 \times 2.30 = (10.72, 19.28)$$

- Current mean response: Not for future. Use contrasts formula.

$$s_{\hat{y}_1} = \sqrt{\frac{s_e^2 \sum h_i^2}{2^2 r}} = \sqrt{\frac{12.75 \times 4}{12}} = 2.06$$

- 90% confidence interval:

$$15 \mp 1.86 \times 2.06 = (11.17, 18.83)$$

# Homework 18A

**Updated** Exercise 18.1: The following table lists measured CPU times for two processors on two workloads. Each experiment was repeated three times. Determine the effects.

**Table 18.12**  $2^2$  3 Experimental Design Exercise

Workload	Processor	
	A	B
I	( 41.16, 39.02, 42.56)	( 65.17, 69.25, 64.23)
J	( 53.50, 55.50, 50.50)	( 50.08, 48.98, 47.10)

# Assumptions

1. Errors are statistically independent.
2. Errors are additive.
3. Errors are normally distributed.
4. Errors have a constant standard deviation  $\sigma_e$ .
5. Effects of factors are additive  
 $\Rightarrow$  observations are independent and normally distributed with constant variance.

# Visual Tests

## 1. Independent Errors:

- ❑ Scatter plot of residuals versus the predicted response  $\hat{y}_i$
- ❑ Magnitude of residuals  $<$  Magnitude of responses/10  
 $\Rightarrow$  Ignore trends
- ❑ Plot the residuals as a function of the experiment number
- ❑ Trend up or down  $\Rightarrow$  other factors or side effects

## 2. Normally distributed errors:

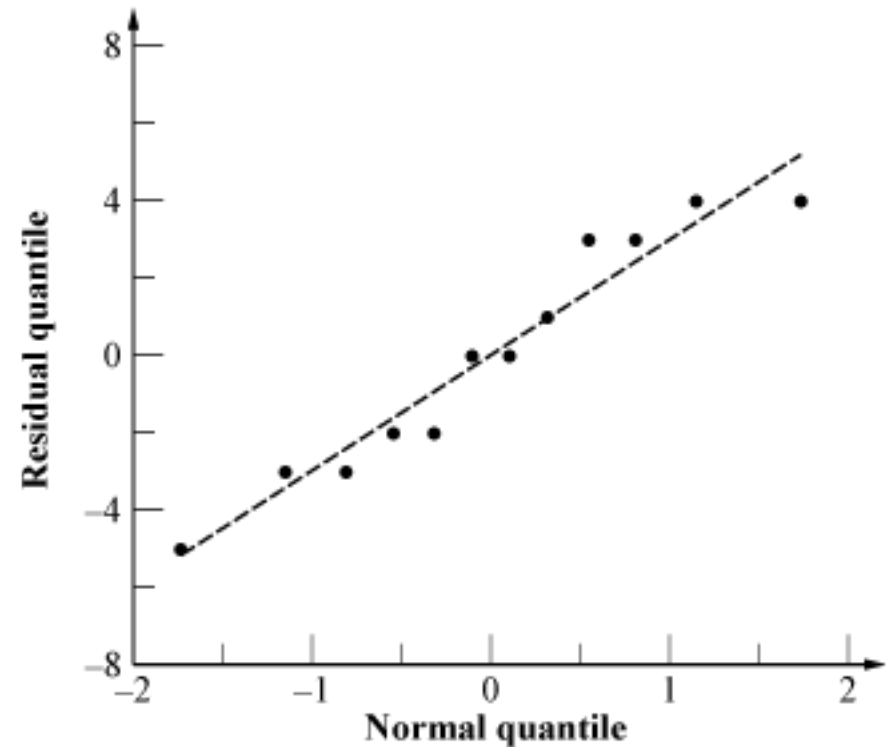
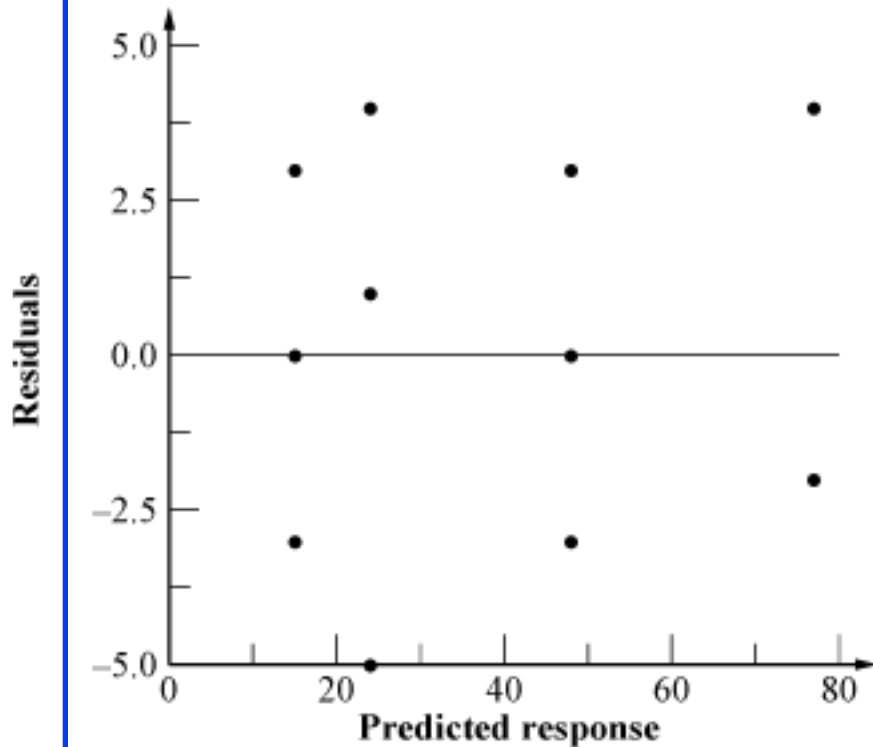
Normal quantile-quantile plot of errors

## 3. Constant Standard Deviation of Errors:

Scatter plot of  $y$  for various levels of the factor

Spread at one level significantly different than that at other  
 $\Rightarrow$  Need transformation

# Example 18.7: Memory-cache



# Multiplicative Models

- ❑ Additive model:

$$y_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij}$$

- ❑ Not valid if effects do not add.

E.g., execution time of workloads.

$i$ th processor speed =  $v_i$  instructions/second.

$j$ th workload Size =  $w_j$  instructions

- ❑ The two effects multiply. Logarithm  $\Rightarrow$  additive model:

$$\text{Execution Time } y_{ij} = w_j / v_i$$

$$\log(y_{ij}) = \log(w_j) - \log(v_i)$$

- ❑ Correct Model:

$$y'_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij}$$

Where,  $y'_{ij} = \log(y_{ij})$



# Multiplicative Model (Cont)

- Taking an antilog of effects:

$$u_A = 10^{q_A}, u_B = 10^{q_B}, \text{ and } u_{AB} = 10^{q_{AB}}$$

- $u_A$  = ratio of MIPS rating of the two processors
- $u_B$  = ratio of the size of the two workloads.
- Antilog of additive mean  $q_0 \Rightarrow$  geometric mean

$$\dot{y} = 10^{q_0} = (y_1 y_2 \cdots y_n)^{1/n} \quad n = 2^r$$

# Example 18.8: Execution Times

Analysis Using an Additive Model

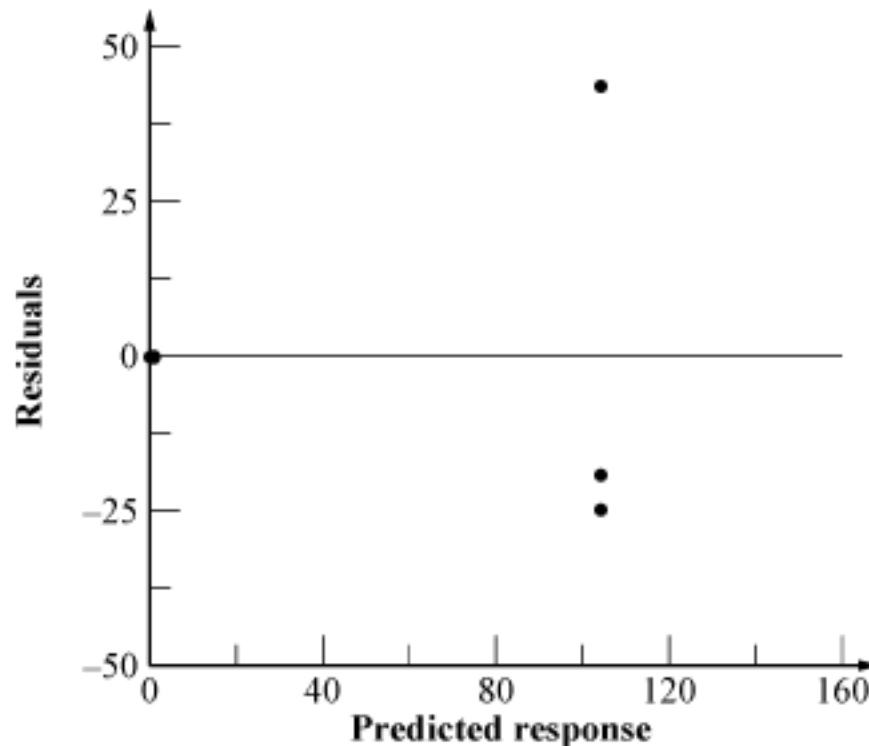
I	A	B	AB	y	Mean $\bar{y}$
1	-1	-1	1	( 85.10, 79.50, 147.90)	104.170
1	1	-1	-1	( 0.891, 1.047, 1.072)	1.003
1	-1	1	-1	( 0.955, 0.933, 1.122)	1.003
1	1	1	1	( 0.0148, 0.0126, 0.0118)	0.013
106.19	-104.15	-104.15	102.17	total	
26.55	-26.04	-26.04	25.54	total/4	

Additive model is not valid because:

- ❑ Physical consideration  $\Rightarrow$  effects of workload and processors do not add. They multiply.
- ❑ Large range for y.  $y_{\max}/y_{\min} = 147.90/0.0118$  or 12,534  $\Rightarrow$  log transformation
- ❑ Taking an arithmetic mean of 114.17 and 0.013 is inappropriate.

## Example 18.8 (Cont)

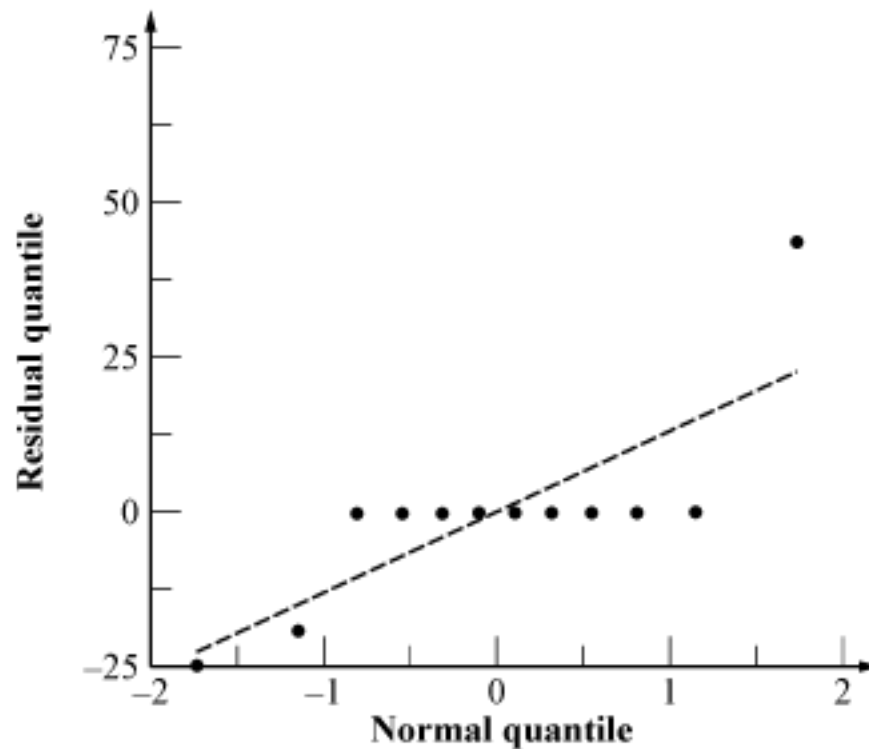
- The residuals are not small as compared to the response.



- The spread of residuals is large at larger value of the response.  
⇒ log transformation

## Example 18.8 (Cont)

- Residual distribution has a longer tail than normal



# Analysis Using Multiplicative Model

## □ Data After Log Transformation

I	A	B	AB	y	Mean $\bar{y}$
1	-1	-1	1	( 1.93, 1.90, 2.17)	2.00
1	1	-1	-1	( -0.05, 0.02, 0.03)	0.00
1	-1	1	-1	( -0.02, -0.03, 0.05)	0.00
1	1	1	1	( -1.83, -1.90, -1.93)	-1.89
0.11	-3.89	-3.89	0.11	total	
0.03	-0.97	-0.97	0.03	total/4	

# Variation Explained by the Two Models

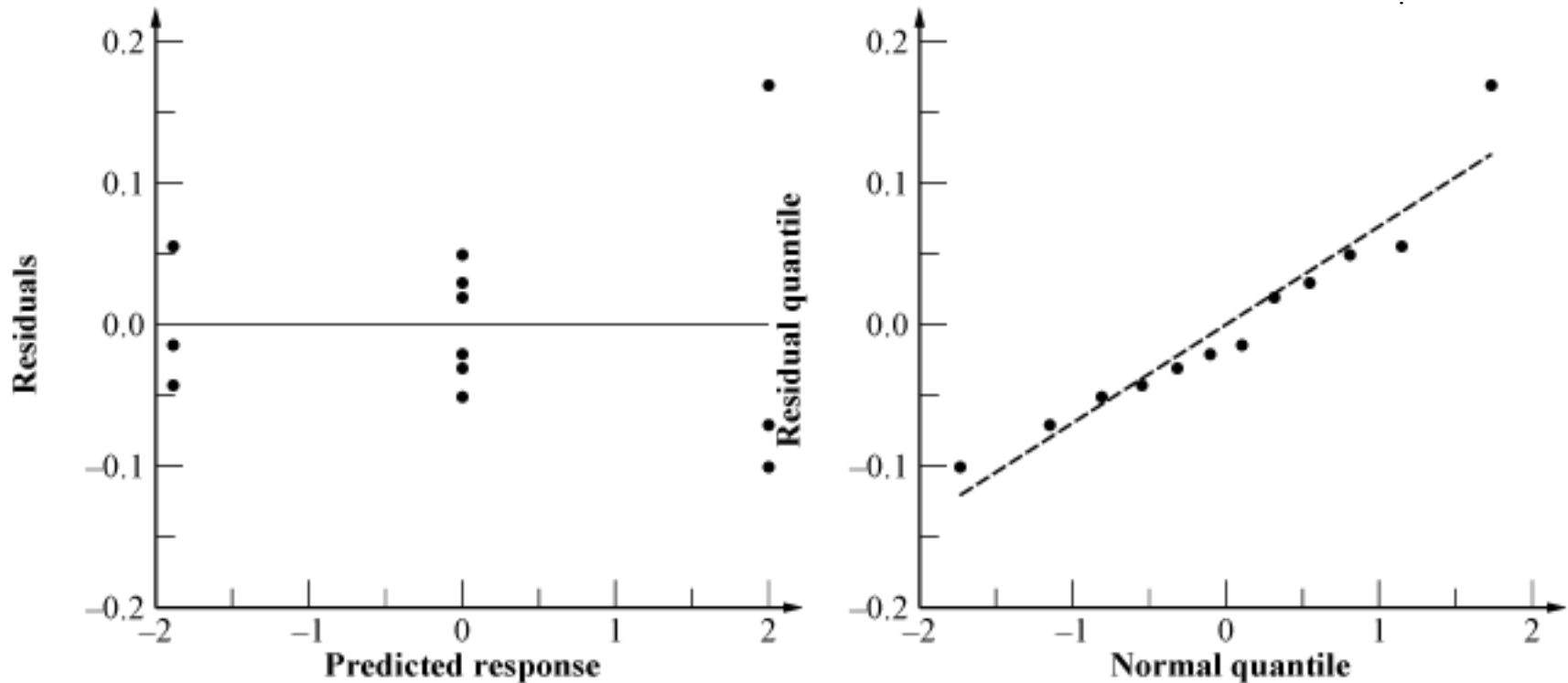
Factor	Additive Model			Multiplicative Model		
	Effect	% Var.	Conf. Interval	Effect	% Var.	Conf. Interval
I	26.55		( 16.35, 36.74)	0.03		( -0.02, 0.07)†
A	-26.04	30.1%	( -36.23, -15.84)	-0.97	49.9%	( -1.02, -0.93)
B	-26.04	30.1%	( -36.23, -15.84)	-0.97	49.9%	( -1.02, -0.93)
AB	25.54	29.0%	( 15.35, 35.74)	0.03	0.0%	( -0.02, 0.07)†
e		10.8%			0.2%	

† ⇒ Not Significant

## □ With multiplicative model:

- Interaction is almost zero.
- Unexplained variation is only 0.2%

# Visual Tests



❑ **Conclusion:** Multiplicative model is better than the additive model.

# Interpretation of Results

$$\log(y) = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e$$

$$\begin{aligned}\Rightarrow y &= 10^{q_0} 10^{q_A x_A} 10^{q_B x_B} 10^{q_{AB} x_A x_B} 10^e \\ &= 10^{0.03} 10^{-0.97 x_A} 10^{-0.97 x_B} 10^{0.03 x_A x_B} 10^e \\ &= 1.07 \times 0.107^{x_A} \times 0.107^{x_B} \times 1.07^{x_A x_B} 10^e\end{aligned}$$

- ❑ The time for an average processor on an average benchmark is 1.07.
- ❑ The time on processor  $A_1$  is 9.35 times ( $0.107^{-1}$ ) that on an average processor. The time on  $A_2$  is  $1/9.35$  ( $0.107^1$ ) of that on an average processor.
- ❑ MIPS rate for  $A_2$  is 87.4 times that of  $A_1$ .
- ❑ Benchmark  $B_1$  executes 87.4 times more instructions than  $B_2$ .
- ❑ The interaction is negligible.

$\Rightarrow$  Results apply to all benchmarks and processors.



# Transformation Considerations

- ❑  $y_{\max}/y_{\min}$  small  $\Rightarrow$  Multiplicative model results similar to additive model.

- ❑ Many other transformations possible.

- ❑ Box-Cox family of transformations:

$$w = \begin{cases} \frac{y^a - 1}{ag^{a-1}}, & a \neq 0 \\ (\ln y)g, & a = 0 \end{cases}$$

- ❑ Where  $g$  is the geometric mean of the responses:

$$g = (y_1 y_2 \cdots y_n)^{1/n}$$

- ❑  $w$  has the same units as  $y$ .

- ❑  $a$  can have any real value, positive, negative, or zero.

- ❑ Plot SSE as a function of  $a \Rightarrow$  optimal  $a$

- ❑ Knowledge about the system behavior should always take precedence over statistical considerations.

# General $2^k$ Factorial Design

## □ Model:

$$y_{ij} = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi} + \cdots + e_{ij}$$

## □ Parameter estimation:

$$q_j = \frac{1}{2^k} \sum_{i=1}^{2^k} S_{ij} \bar{y}_i$$

$S_{ij}$  =  $(i,j)$ th entry in the sign table.

## □ Sum of squares:

$$SSY = \sum_{i=1}^{2^k} \sum_{j=1}^r y_{ij}^2$$

$$SS0 = 2^k r q_0^2$$

$$SST = SSY - SS0$$

$$SS_j = 2^k r q_j^2 \quad j = 1, 2, \dots, 2^k - 1$$

$$SSE = SST - \sum_{j=1}^{2^k - 1} SS_j$$

# General $2^k r$ Factorial Design (Cont)

- Percentage of  $y$ 's variation explained by  $j$ th effect =

$$(SS_j/SST) \times 100\%$$

- Standard deviation of errors:

$$s_e = \sqrt{\frac{SSE}{2^k(r-1)}}$$

- Standard deviation of effects:

$$s_{q_0} = s_{q_A} = s_{q_B} = s_{q_{AB}} = s_e / \sqrt{2^k r}$$

- Variance of contrast  $\sum h_i q_i$ , where  $\sum h_i = 0$  is:

$$s_{\sum h_i q_i}^2 = (s_e^2 \sum h_i^2) / 2^k r$$

# General $2^k r$ Factorial Design (Cont)

- Standard deviation of the mean of  $m$  future responses:

$$s_{\hat{y}_p} = s_e \left( \frac{1 + 2^k}{2^k r} + \frac{1}{m} \right)^{1/2}$$

- Confidence intervals are calculated using  $t_{[1-\alpha/2; 2^k(r-1)]}$ .
- Modeling assumptions:
  - Errors are IID normal variates with zero mean.
  - Errors have the same variance for all values of the predictors.
  - Effects and errors are additive.

# Visual Tests for $2^k$ Designs

- ❑ The scatter plot of errors versus predicted responses should not have any trend.
- ❑ The normal quantile-quantile plot of errors should be linear.
- ❑ Spread of  $y$  values in all experiments should be comparable.

# Example 18.9: A 2<sup>3</sup> Design

I	A	B	C	A B	A C	B C	A B C	y	Mean $\bar{y}$
1	-1	-1	-1	1	1	1	-1	(14, 16, 12)	14
1	1	-1	-1	-1	-1	1	1	(22, 18, 20)	20
1	-1	1	-1	-1	1	-1	1	(11, 15, 19)	15
1	1	1	-1	1	-1	-1	-1	(34, 30, 35)	33
1	-1	-1	1	1	-1	-1	1	(46, 42, 44)	44
1	1	-1	1	-1	1	-1	-1	(58, 62, 60)	60
1	-1	1	1	-1	-1	1	-1	(50, 55, 54)	53
1	1	1	1	1	1	1	1	(86, 80, 74)	80
319	67	43	155	23	19	15	-1	total	
39.87	8.375	5.375	19.37	2.875	2.375	1.875	-0.125	total/8	

## Example 18.9 (Cont)

### □ Sum of Squares:

Component	Sum of Squares	Percent Variation
$y$	4.9E4	
$\bar{y}$	3.8E4	
$y-\bar{y}$	1.1E4	100.00%
A	1683.0	14.06%
B	693.3	5.79%
C	9009.0	75.27%
AB	198.3	1.66%
AC	135.4	1.13%
BC	84.4	0.70%
ABC	0.4	0.00%
Errors	164.0	1.37%

## Example 18.9 (Cont)

- The errors have  $2^3(3-1)$  or 16 degrees of freedom. Standard deviation of errors:

$$s_e = \sqrt{\frac{\text{SSE}}{2^k(r-1)}} = \sqrt{\frac{164}{16}} = 3.20$$

- Standard deviation of effects:

$$s_{q_i} = s_e / \sqrt{(2^3 3)} = 3.20 / \sqrt{24} = 0.654$$



## Example 18.9 (Cont)

### □ % Variation:

Component	Sum of Squares	Percent Variation
$y$	4.9E4	
$\bar{y}$	3.8E4	
$y-\bar{y}$	1.1E4	100.00%
A	1683.0	14.06%
B	693.3	5.79%
C	9009.0	75.27%
AB	198.3	1.66%
AC	135.4	1.13%
BC	84.4	0.70%
ABC	0.4	0.00%
Errors	164.0	1.37%

## Example 18.9 (Cont)

- $t_{[0.90, 16]} = 1.337$
- 80% confidence intervals for parameters:  $q_i \mp (1.337)(0.654)$   
 $= q_i \mp 0.874$

$$q_0 = (39.00, 40.74)$$

$$q_A = (7.50, 9.25)$$

$$q_B = (4.50, 6.25)$$

$$q_C = (18.50, 20.24)$$

$$q_{AB} = (2.00, 3.75)$$

$$q_{AC} = (1.50, 3.25)$$

$$q_{BC} = (1.00, 2.75)$$

$$q_{ABC} = (-1.00, 0.75)$$

- All effects except  $q_{ABC}$  are significant.

## Example 18.9 (Cont)

- For a single confirmation experiment ( $m = 1$ )

With  $A = B = C = -1$ :

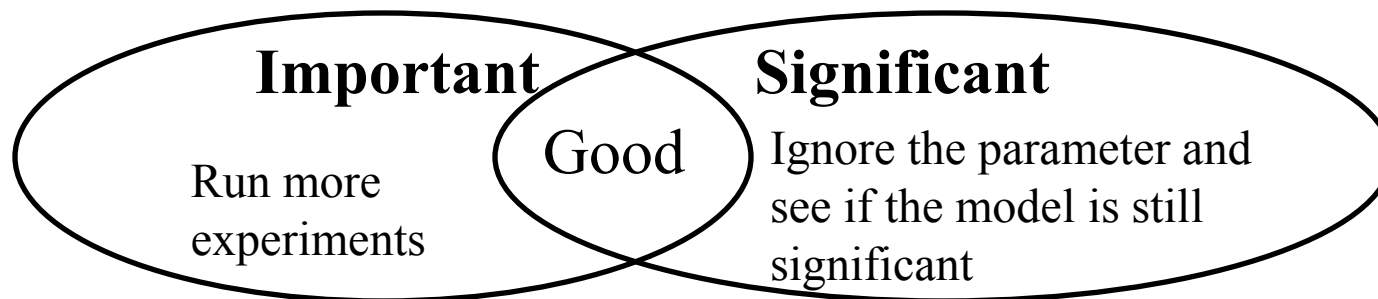
$$\begin{aligned}\hat{y} &= 14 \\ s_{\hat{y}} &= s_e \left( \frac{9}{2k_r} + \frac{1}{m} \right)^{1/2} \\ &= 3.2 \left( \frac{9}{24} + 1 \right)^{1/2} \\ &= 3.75\end{aligned}$$

- 80% confidence interval:

$$14 \mp 1.337 \times 3.75 = 14 \mp 5.02 = (8.98, 19.02)$$

# Importance vs. Significance

- ❑ Important = Parameters or models that explain a high percent of variation
- ❑ Significant = Parameters or models whose confidence interval does not include zero



# Case Study 18.1: Garbage collection

## Factors and Levels

Variable	Factor	-1	1
A	Workload	Single Task	Several parallel tasks
B	Compiler	Simple	Deallocating
C	Limbo List	Enabled	Disabled
D	Chunk Size	4K bytes	16K bytes

# Case Study 18.1 (Cont)

I	A	B	C	D	y	Mean $\bar{y}$
1	-1	-1	-1	-1	( 97, 97, 97)	97.00
1	1	-1	-1	-1	( 31, 31, 32)	31.33
1	-1	1	-1	-1	( 97, 97, 97)	97.00
1	1	1	-1	-1	( 31, 32, 31)	31.33
1	-1	-1	1	-1	( 97, 97, 97)	97.00
1	1	-1	1	-1	( 32, 32, 31)	31.67
1	-1	1	1	-1	( 97, 97, 97)	97.00
1	1	1	1	-1	( 32, 32, 32)	32.00
1	-1	-1	-1	1	( 407, 407, 407)	407.00
1	1	-1	-1	1	( 135, 136, 135)	135.33
1	-1	1	-1	1	( 409, 409, 409)	409.00
1	1	1	-1	1	( 135, 135, 136)	135.33
1	-1	-1	1	1	( 407, 407, 407)	407.00
1	1	-1	1	1	( 139, 140, 139)	139.33
1	-1	1	1	1	( 409, 409, 409)	409.00
1	1	1	1	1	( 139, 139, 140)	139.33
2695.67	-1344.33	4.33	9.00	1667.00		total
168.48	-84.02	0.27	0.56	104.19		total/16

# Case Study 18.1 (Cont)

Factor	Effect	% Variation	Conf. Interval
I	168.48		( 168.386, 168.573)
A	-84.02	34.4%	( -84.114, -83.927)
B	0.27	0.0%	( 0.177, 0.364)
C	0.56	0.0%	( 0.469, 0.656)
D	104.19	52.8%	( 104.094, 104.281)
AB	-0.23	0.0%	( -0.323, -0.136)
AC	0.56	0.0%	( 0.469, 0.656)
AD	-51.31	12.8%	( -51.406, -51.219)
BC	0.02	0.0%	( -0.073, 0.114)†
BD	0.23	0.0%	( 0.136, 0.323)
CD	0.44	0.0%	( 0.344, 0.531)
ABC	0.02	0.0%	( -0.073, 0.114)†
ABD	-0.27	0.0%	( -0.364, -0.177)
ACD	0.44	0.0%	( 0.344, 0.531)
BCD	-0.02	0.0%	( -0.114, 0.073)†
ABCD	-0.02	0.0%	( -0.114, 0.073)†

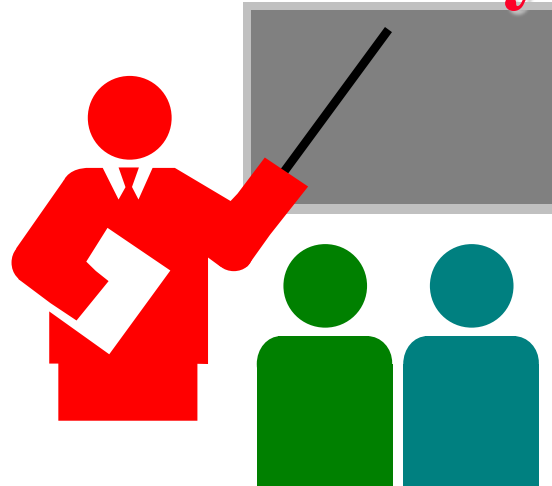
†  $\Rightarrow$  Not Significant

# Case Study 18.1: Conclusions

- ❑ Most of the variation is explained by factors A (Workload), D (Chunk size), and the interaction A D between the two.
- ❑ The variation due to experimental error is small  
⇒ Several effects that explain less than 0.05% of variation (listed as 0.0%) are statistically significant.
- ❑ Only effects A, D, and AD are both practically significant and statistically significant.



# Summary



- ❑ Replications allow estimation of measurement errors
  - ⇒ Confidence Intervals of parameters
  - ⇒ Confidence Intervals of predicted responses
- ❑ Allocation of variation is proportional to square of effects
- ❑ Multiplicative models are appropriate if the factors multiply
- ❑ Visual tests for independence normal errors

# Homework 18B

**Updated** Exercise 18.1: For the data of Homework 18A, determine percentage of variation explained, find confidence intervals of the effects, and conduct visual tests.