# $2^{k-p}$ Fractional Factorial Designs

Raj Jain
Washington University in Saint Louis
Saint Louis, MO 63130
Jain@cse.wustl.edu

These slides are available on-line at:

http://www.cse.wustl.edu/~jain/cse567-15/


- $2^{k-p}$ Fractional Factorial Designs
- Sign Table for a $2^{k-p}$ Design
- Confounding
- Other Fractional Factorial Designs
- Algebra of Confounding
- Design Resolution

# $2^{k-p}$ Fractional Factorial Designs

- Large number of factors
  ⇒ large number of experiments
  ⇒ full factorial design too expensive
  ⇒ Use a fractional factorial design
- $2^{k-p}$ design allows analyzing k factors with only $2^{k-p}$ experiments.
  $2^{k-1}$ design requires only half as many experiments
  $2^{k-2}$ design requires only one quarter of the experiments

# Example: $2^{7-4}$ Design

| Expt No. | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 |
| 2 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| 3 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 4 | 1 | 1 | -1 | 1 | -1 | -1 | -1 |
| 5 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| 6 | 1 | -1 | 1 | -1 | 1 | -1 | -1 |
| 7 | -1 | 1 | 1 | -1 | -1 | 1 | -1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

- Study 7 factors with only 8 experiments!

# Fractional Design Features

- Full factorial design is easy to analyze due to orthogonality of sign vectors.

  Fractional factorial designs also use orthogonal vectors. That is:

  - The sum of each column is zero.

$$\sum_i x_{ij} = 0 \quad \forall j$$

*j*th variable, *i*th experiment.

  - The sum of the products of any two columns is zero.

$$\sum_i x_{ij} x_{il} = 0 \quad \forall j \neq l$$

  - The sum of the squares of each column is $2^{7-4}$, that is, 8.

$$\sum_i x_{ij}^2 = 8 \quad \forall j$$

# Analysis of Frac. Factorial Designs

- **Model**:

$$y = q_0 + q_A x_A + q_B x_B + q_C x_C + q_D x_D + q_E x_E + q_F x_F + q_G x_G$$

- Effects can be computed using inner products.

$$q_A = \sum_i y_i x_{Ai} = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}$$

$$q_B = \sum_i y_i x_{Bi} = \frac{-y_1 - y_2 + y_3 + y_4 - y_5 - y_6 + y_7 + y_8}{8}$$

# Example 19.1

| I | A | B | C | D | E | F | G | y |
|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 | 20 |
| 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 35 |
| 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 | 7 |
| 1 | 1 | 1 | -1 | 1 | -1 | -1 | -1 | 42 |
| 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 36 |
| 1 | 1 | -1 | 1 | -1 | 1 | -1 | -1 | 50 |
| 1 | -1 | 1 | 1 | -1 | -1 | 1 | -1 | 45 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 82 |
| 317 | 101 | 35 | 109 | 43 | 1 | 47 | 3 | Total |
| 39.62 | 12.62 | 4.37 | 13.62 | 5.37 | 0.125 | 5.87 | 0.37 | Total/8 |

- Factors A through G explain 37.26%, 4.47%, 43.40%, 6.75%, 0%, 8.06%, and 0.03% of variation, respectively.

  ⇒ Use only factors C and A for further experimentation.

# Sign Table for a $2^{k-p}$ Design

Steps:

1. Prepare a sign table for a full factorial design with k-p factors.
2. Mark the first column I.
3. Mark the next k-p columns with the k-p factors.
4. Of the $(2^{k-p}-k+p-1)$ columns on the right, choose p columns and mark them with the p factors which were not chosen in step 1.

# Example: $2^{7-4}$ Design

| Expt No. | A | B | C | AB | AC | BC | ABC |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 |
| 2 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| 3 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 4 | 1 | 1 | -1 | 1 | -1 | -1 | -1 |
| 5 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| 6 | 1 | -1 | 1 | -1 | 1 | -1 | -1 |
| 7 | -1 | 1 | 1 | -1 | -1 | 1 | -1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

# Example: $2^{4-1}$ Design

| Expt No. | A | B | C | AB | AC | BC | D |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 |
| 2 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| 3 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 4 | 1 | 1 | -1 | 1 | -1 | -1 | -1 |
| 5 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| 6 | 1 | -1 | 1 | -1 | 1 | -1 | -1 |
| 7 | -1 | 1 | 1 | -1 | -1 | 1 | -1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

# Confounding

- **Confounding**: Only the combined influence of two or more effects can be computed.

$$q_A = \sum_i y_i x_{Ai} = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}$$

$$q_D = \sum_i y_i x_{Di} = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

# Confounding (Cont)

$$q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci}$$

$$= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

$$q_D = q_{ABC}$$

$$q_D + q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci}$$

$$= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

- ⇒ Effects of D and ABC are confounded. Not a problem if $q_{ABC}$ is negligible.

# Confounding (Cont)

- Confounding representation: *D=ABC*

  Other Confoundings:

$$q_A = q_{BCD} = \sum_i y_i x_{Ai}$$

$$= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}$$

$$\Rightarrow A = BCD$$

A=BCD, B=ACD, C=ABD, AB=CD, AC=BD, BC=AD, ABC=D, and I=ABCD

- *I=ABCD* ⇒ confounding of ABCD with the mean.

# Other Fractional Factorial Designs

- A fractional factorial design is not unique. $2^p$ different designs.

Another $2^{4-1}$ Experimental Design

| Expt No. | A | B | C | D | AC | BC | ABC |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 |
| 2 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| 3 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 4 | 1 | 1 | -1 | 1 | -1 | -1 | -1 |
| 5 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| 6 | 1 | -1 | 1 | -1 | 1 | -1 | -1 |
| 7 | -1 | 1 | 1 | -1 | -1 | 1 | -1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

- Confoundings: I=ABD, A=BD, B=AD, C=ABCD, D=AB, AC=BCD, BC=ACD, ABC=CD

Not as good as the previous design.

# Algebra of Confounding

- Given just one confounding, it is possible to list all other confoundings.
- Rules:
  - $I$ is treated as unity.
  - Any term with a power of 2 is erased.

$$I = ABCD$$

Multiplying both sides by A:

$$A = A^2BCD = BCD$$

Multiplying both sides by B, C, D, and AB:

# Algebra of Confounding (Cont)

$$B = AB^2CD = ACD$$

$$C = ABC^2D = ABD$$

$$D = ABCD^2 = ABC$$

$$AB = A^2B^2CD = CD$$

and so on.

- Generator polynomial: $I=ABCD$

For the second design: $I=ABC$.

- In a $2^{k-p}$ design, $2^p$ effects are confounded together.

# Example 19.7

- In the $2^{7-4}$ design:

$$D = AB, E = AC, F = BC, G = ABC$$

$$\Rightarrow I = ABD, I = ACE, I = BCF, I = ABCG$$

$$\Rightarrow I = ABD = ACE = BCF = ABCG$$

- Using products of all subsets:

$$\begin{aligned} I &= ABD = ACE = BCF = ABCG = BCDE \\ &= ACDF = CDG = ABEF = BEG \\ &= AFG = DEF = ADEG = BDFG \\ &= CEFG = ABCDEFG \end{aligned}$$

# Example 19.7 (Cont)

- Other confoundings:

$$\begin{aligned} A &= BD = CE = ABCF = BCG = ABCDE \\ &= CDF = ACDG = BEF = ABEG \\ &= FG = ADEF = DEG = ABDFG \\ &= ACEFG = BCDEFG \end{aligned}$$

# Design Resolution

- Order of an effect = Number of terms
  Order of *ABCD* = 4, order of I = 0.
- Order of a confounding = Sum of order of two terms
  E.g., AB=CDE is of order 5.
- Resolution of a Design
  = Minimum of orders of confoundings
- Notation: $R_{III}$ = Resolution-III = $2^{k-p}_{III}$
- Example 1: *I=ABCD* ⇒ $R_{IV}$ = Resolution-IV = $2^{4-1}_{IV}$
  A=BCD, B=ACD, C=ABD, AB=CD, AC=BD,
  BC=AD, ABC=D, and I=ABCD

# Design Resolution (Cont)

- Example 2:

  $I = ABD \Rightarrow \mathrm{R_{III}}$ design.

- Example 3:

$$\begin{aligned} I &= ABD = ACE = BCF = ABCG = BCDE \\ &= ACDF = CDG = ABEF = BEG \\ &= AFG = DEF = ADEG = BDFG \\ &= CEFG = ABCDEFG \end{aligned}$$

- This is a resolution-III design.
- A design of higher resolution is considered a better design.

# Case Study 19.1: Latex vs. troff

Factors and Levels

| | Factor | -Level | +Level |
|---|---|---|---|
| A | Program | Latex | troff-me |
| B | Bytes | 2100 | 25000 |
| C | Equations | 0 | 10 |
| D | Floats | 0 | 10 |
| E | Tables | 0 | 10 |
| F | Footnotes | 0 | 10 |

# Case Study 19.1 (Cont)

- Design: $2^{6-1}$ with I=BCDEF

| Factor | | Effect | % Variation |
|---|---|---|---|
| B | Bytes | 12.0 | 39.4% |
| A | Program | 9.4 | 24.4% |
| C | Equations | 7.5 | 15.6% |
| AC | Program × Equations | 7.2 | 14.4% |
| E | Tables | 3.5 | 3.4% |
| F | Footnotes | 1.6 | 0.70% |

# Case Study 19.1: Conclusions

- Over 90% of the variation is due to: Bytes, Program, and Equations and a second order interaction.
- Text file size were significantly different making it's effect more than that of the programs.
- High percentage of variation explained by the ``program × Equation" interaction
  ⇒ Choice of the text formatting program depends upon the number of equations in the text. troff not as good for equations.

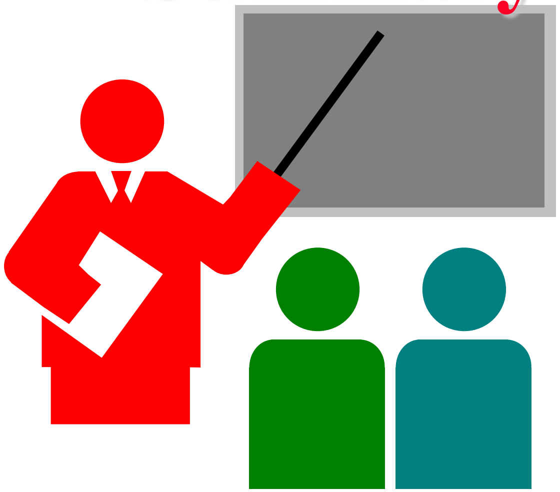
CPU Time

| Program | # of Equations | |
|---|---|---|
| | -1(0) | 1(10) |
| -1(Latex) | -9.7 | -9.1 |
| 1(Troff) | -5.3 | 24.1 |

# Case Study 19.1: Conclusions (Cont)

- Low ``Program $\times$ Bytes'' interaction $\Rightarrow$ Changing the file size affects both programs in a similar manner.
- In next phase, reduce range of file sizes. Alternately, increase the number of levels of file sizes.

# Summary



- Fractional factorial designs allow a large number of variables to be analyzed with a small number of experiments
- Many effects and interactions are confounded
- The resolution of a design is the sum of the order of confounded effects
- A design with higher resolution is considered better

# Exercise 19.1

Analyze the $2^{4-1}$ design:

| | | $C_1$ | | $C_2$ | |
|---|---|---|---|---|---|
| | | $D_1$ | $D_2$ | $D_1$ | $D_2$ |
| $A_1$ | $B_1$ | | 40 | 15 | |
| | $B_2$ | | 20 | 10 | |
| $A_2$ | $B_1$ | 100 | | | 30 |
| | $B_2$ | 120 | | | 50 |

- Quantify all main effects.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.
- List all confoundings.
- Can you propose a better design with the same number of experiments.
- What is the resolution of the design?

# Exercise 19.2

Is it possible to have a $2^{4-1}_{III}$ design? a $2^{4-1}_{II}$ design? $2^{4-1}_{IV}$ design? If yes, give an example.

# Homework 19

- **Updated** Exercise 19.1
  Analyze the $2^{4-1}$ design:

| | | $C_1$ | | $C_2$ | |
|---|---|---|---|---|---|
| | | $D_1$ | $D_2$ | $D_1$ | $D_2$ |
| $A_1$ | $B_1$ | | 30 | 15 | |
| | $B_2$ | | 20 | 10 | |
| $A_2$ | $B_1$ | 100 | | | 30 |
| | $B_2$ | 110 | | | 50 |

- Quantify all main effects.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.
- List all confoundings.
- Can you propose a better design with the same number of experiments.
- What is the resolution of the design?