# Two Factors Full Factorial Design without Replications

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- Computation of Effects
- Estimating Experimental Errors
- Allocation of Variation
- ANOVA Table
- Visual Tests
- Confidence Intervals For Effects
- Multiplicative Models
- Missing Observations

# Two Factors Full Factorial Design

- Used when there are two parameters that are carefully controlled
- Examples:
  - To compare several processors using several workloads.
  - To determining two configuration parameters, such as cache and memory sizes
- Assumes that the factors are categorical. For quantitative factors, use a regression model.
- A full factorial design with two factors *A* and *B* having *a* and *b* levels requires *ab* experiments.
- First consider the case where each experiment is conducted only once.

# Model

$$y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}$$

$$\begin{aligned} y_{ij} &= \text{Observation with A at level j} \\ &\quad \text{and B at level i} \\ \mu &= \text{mean response} \\ \alpha_j &= \text{effect of factor A at level j} \\ \beta_i &= \text{effect of factor B at level i} \\ e_{ij} &= \text{error term} \end{aligned}$$

# Computation of Effects

- Averaging the jth column produces:

$$\bar{y}_{.j} = \mu + \alpha_j + \frac{1}{b}\sum_i \beta_i + \frac{1}{b}\sum_i e_{ij}$$

- Since the last two terms are zero, we have:

$$\bar{y}_{.j} = \mu + \alpha_j$$

- Similarly, averaging along rows produces:

$$\bar{y}_{i.} = \mu + \beta_i$$

- Averaging all observations produces

$$\bar{y}_{..} = \mu$$

- Model parameters estimates are:

$$\begin{aligned} \mu &= \bar{y}_{..} \\ \alpha_j &= \bar{y}_{.j} - \bar{y}_{..} \\ \beta_i &= \bar{y}_{i.} - \bar{y}_{..} \end{aligned}$$

- Easily computed using a tabular arrangement.

# Example 21.1: Cache Comparison

| Workloads | Two Caches | One Cache | No Cache |
|---|---|---|---|
| ASM | 54.0 | 55.0 | 106.0 |
| TECO | 60.0 | 60.0 | 123.0 |
| SIEVE | 43.0 | 43.0 | 120.0 |
| DHRYSTONE | 49.0 | 52.0 | 111.0 |
| SORT | 49.0 | 50.0 | 108.0 |

# Example 21.1: Computation of Effects

| Workloads | Two Caches | One Cache | No Cache | Row Sum | Row Mean | Row Effect |
|---|---|---|---|---|---|---|
| ASM | 54.0 | 55.0 | 106.0 | 215.0 | 71.7 | -0.5 |
| TECO | 60.0 | 60.0 | 123.0 | 243.0 | 81.0 | 8.8 |
| SIEVE | 43.0 | 43.0 | 120.0 | 206.0 | 68.7 | -3.5 |
| DHRYSTONE | 49.0 | 52.0 | 111.0 | 212.0 | 70.7 | -1.5 |
| SORT | 49.0 | 50.0 | 108.0 | 207.0 | 69.0 | -3.2 |
| Column Sum | 255.0 | 260.0 | 568.0 | 1083.0 | | |
| Column Mean | 51.0 | 52.0 | 113.6 | | 72.2 | |
| Column effect | -21.2 | -20.2 | 41.4 | | | |

- An average workload on an average processor requires 72.2 ms of processor time.
- The time with two caches is 21.2 ms lower than that on an average processor
- The time with one cache is 20.2 ms lower than that on an average processor.
- The time without a cache is 41.4 ms higher than the average

# Example 21.1 (Cont)

- Two-cache - One-cache = 1 ms.
- One-cache - No-cache = 41.4+20.2 or 61.6 ms.
- The workloads also affect the processor time required.
- The ASM workload takes  0.5 ms less than the average.
- TECO takes 8.8 ms higher than the average.

# Estimating Experimental Errors

- Estimated response:

$$\hat{y}_{ij} = \mu + \alpha_j + \beta_i$$

- Experimental error:

$$e_{ij} = y_{ij} - \hat{y}_{ij} = y_{ij} - \mu - \alpha_j - \beta_i$$

- Sum of squared errors (SSE):

$$\text{SSE} = \sum_{i=1}^{b} \sum_{j=1}^{a} e_{ij}^2$$

- Example: The estimated processor time is:

$$\hat{y}_{11} = \mu + \alpha_1 + \beta_1 = 72.2 - 21.2 - 0.5 = 50.5$$

- Error = Measured-Estimated = 54-50.5 = 3.5

# Example 21.2: Error Computation

| Workloads | Two Caches | One Cache | No Cache |
|---|---|---|---|
| ASM | 3.5 | 3.5 | -7.1 |
| TECO | 0.2 | -0.8 | 0.6 |
| SIEVE | -4.5 | -5.5 | 9.9 |
| DHRYSTONE | -0.5 | 1.5 | -1.1 |
| SORT | 1.2 | 1.2 | -2.4 |

The sum of squared errors is:

$$\text{SSE} = (3.5)^2 + (0.2)^2 + \cdots + (-2.4)^2 = 236.80$$

# Example 21.2: Allocation of Variation

$$y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}$$

- Squaring the model equation:

$$\begin{array}{ccccccccc} \sum_{ij} y_{ij}^2 & = & ab\mu^2 & + & b\sum_j \alpha_j^2 & + & a\sum_i \beta_i^2 & + & \sum_{ij} e_{ij}^2 \\ \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \end{array}$$

$$\begin{array}{ccccccccccc} \text{SST} & = & \text{SSY} & - & \text{SS0} & = & \text{SSA} & + & \text{SSB} & + & \text{SSE} \\ 13402.41 & = & 91595 & - & 78192.59 & = & 12857.20 & + & 308.40 & + & 236.80 \\ 100\% & = & & & & = & 95.9\% & + & 2.3\% & + & 1.8\% \\ ab-1 & = & ab & - & 1 & = & (a-1) & + & (b-1) & + & (a-1)(b-1) \end{array}$$

- High percent variation explained
  ⇒ Cache choice <u>important</u> in processor design.

# Analysis of Variance

- Degrees of freedoms:

$$\begin{array}{ccccccccc} \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \\ ab & = & 1 & + & (a-1) & + & (b-1) & + & (a-1)(b-1) \end{array}$$

- Mean squares:

$$\text{MSA} = \frac{\text{SSA}}{a-1}$$

$$\text{MSB} = \frac{\text{SSB}}{b-1}$$

$$\text{MSE} = \frac{\text{SSE}}{(a-1)(b-1)}$$

$$\frac{MSA}{MSE} \sim F_{[1-\alpha;a-1,(a-1)(b-1)]}$$

- Computed ratio > $F_{[1-\alpha;a-1,(a-1)(b-1)]}$ $\Rightarrow$ A is significant at level $\alpha$.

# ANOVA Table

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| $y$ | $\text{SSY} = \sum y_{ij}^2$ | | $ab$ | | | |
| $\bar{y}_{..}$ | $\text{SS0} = ab\mu^2$ | | 1 | | | |
| $y - \bar{y}_{..}$ | SST=SSY-SS0 | 100 | $ab-1$ | | | |
| $A$ | $\text{SSA} = b\Sigma\alpha_j^2$ | $100\left(\frac{\text{SSA}}{\text{SST}}\right)$ | $a-1$ | $\text{MSA} = \frac{\text{SSA}}{a-1}$ | $\frac{\text{MSA}}{\text{MSE}}$ | $F_{[1-\alpha, a-1, (a-1)(b-1)]}$ |
| $B$ | $\text{SSB} = a\Sigma\beta_i^2$ | $100\left(\frac{\text{SSB}}{\text{SST}}\right)$ | $b-1$ | $\text{MSB} = \frac{\text{SSB}}{b-1}$ | $\frac{\text{MSB}}{\text{MSE}}$ | $F_{[1-\alpha, b-1, (a-1)(b-1)]}$ |
| $e$ | $\text{SSE} = \text{SST} - (\text{SSA} + \text{SSB})$ | $100\left(\frac{\text{SSE}}{\text{SST}}\right)$ | $(a-1)(b-1)$ | $\text{MSE} = \frac{\text{SSE}}{(a-1)(b-1)}$ | | |

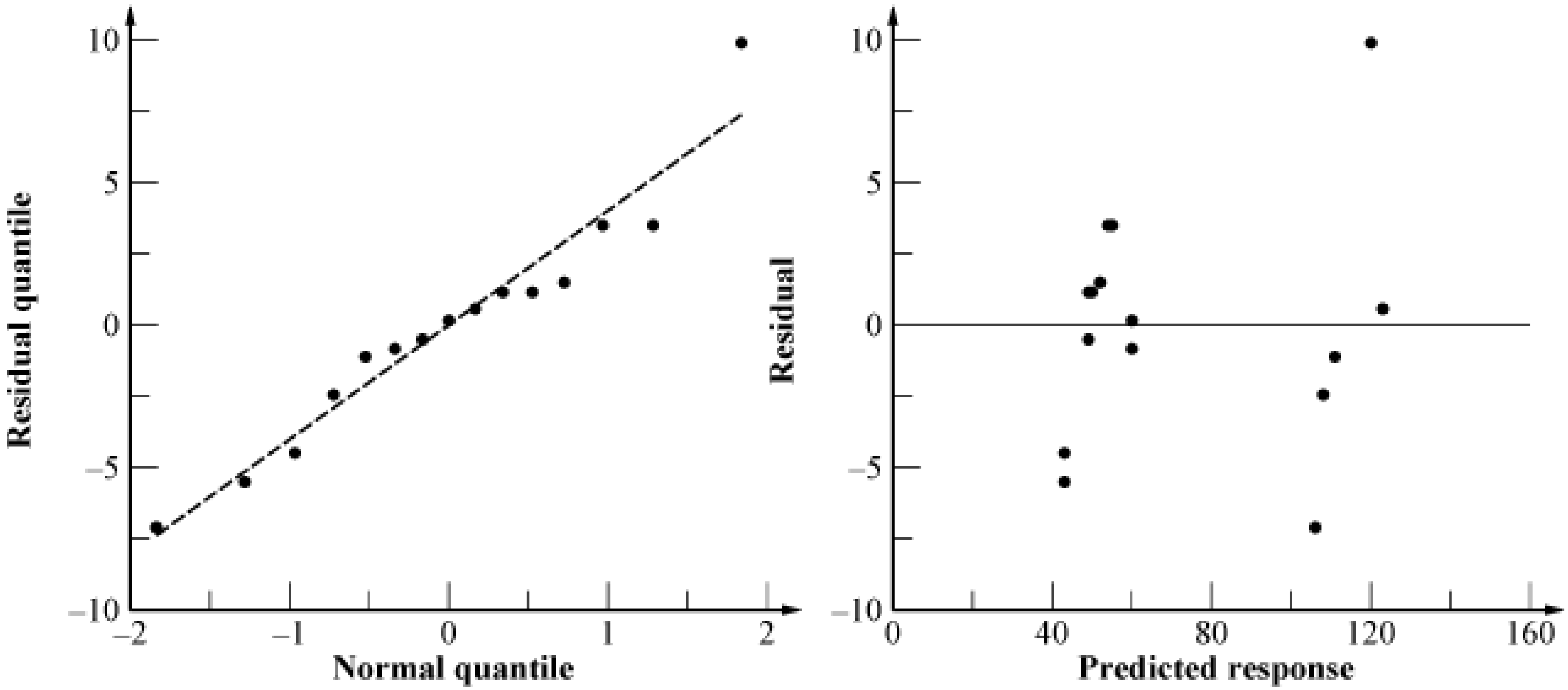
# Example 21.3: Cache Comparison

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 91595.00 | | | | | |
| $\bar{y}_{..}$ | 78192.59 | | | | | |
| y-$\bar{y}_{..}$ | 13402.41 | 100.0% | 14 | | | |
| Caches | 12857.20 | 95.9% | 2 | 6428.60 | 217.2 | 3.1 |
| Workloads | 308.40 | 2.3% | 4 | 77.10 | 2.6 | 2.8 |
| Errors | 236.80 | 1.8% | 8 | 29.60 | | |

$$s_e = \sqrt{\text{MSE}} = \sqrt{29.60} = 5.44$$

- Cache choice significant.
- Workloads insignificant

# Example 21.4: Visual Tests



# Confidence Intervals For Effects

| Parameter | Estimate | Variance |
|---|---|---|
| $\mu$ | $\bar{y}_{..}$ | $s_e^2/ab$ |
| $\alpha_j$ | $\bar{y}_{.j}$-$\bar{y}_{..}$ | $s_e^2(a-1)/ab$ |
| $\mu+\alpha_j$ | $\bar{y}_{.j}$ | $s_e^2/b$ |
| $\beta_i$ | $\bar{y}_{i.}$-$\bar{y}_{..}$ | $s_e^2(b-1)/ab$ |
| $\mu+\alpha_j+\beta_i$ | $\bar{y}_{.j}+\bar{y}_{i.}$-$\bar{y}_{..}$ | $s_e^2(a+b-1)/(ab)$ |
| $\sum_{j=1}^a h_j\ \alpha_j,\ \sum_{j=1}^a h_j = 0$ | $\sum_{j=1}^a h_j\ \bar{y}_{.j}$ | $s_e^2 \sum_{j=1}^a h_j^2/b$ |
| $\sum_{i=1}^b h_i\ \beta_i,\ \sum_{i=1}^b h_i = 0$ | $\sum_{i=1}^b h_i\ \bar{y}_{i.}$ | $s_e^2 \sum_{i=1}^b h_i^2/a$ |
| $s_e^2$ | $\{\sum_{j=1}^a \sum_{i=1}^b e_{ij}^2\}/\{(a-1)(b-1)\}$ | |

Degrees of freedom for errors = (a-1)(b-1)

- For confidence intervals use $t$ values at ($a$-1)($b$-1) degrees of freedom

# Example 21.5: Cache Comparison

- Standard deviation of errors:

$$s_e = \sqrt{\text{MSE}} = \sqrt{29.60} = 5.4$$

- Standard deviation of the grand mean:

$$s_\mu = s_e/\sqrt{ab} = 5.4/\sqrt{15} = 1.4$$

- Standard deviation of $\alpha_j$'s:

$$s_{\alpha_j} = s_e\sqrt{(a-1)/ab} = 5.4\sqrt{\frac{2}{15}} = 2.0$$

- Standard deviation of $\beta_i$'s:

$$s_{\beta_i} = s_e\sqrt{(b-1)/ab} = 5.4\sqrt{\frac{4}{15}} = 2.8$$

# Example 21.5 (Cont)

- Degrees of freedom for the errors are (a-1)(b-1)=8. For 90% confidence interval, $t_{[0.95;8]}$= 1.86.
- Confidence interval for the grand mean:

$$72.2 \mp 1.86 \times 1.4 = 72.2 \mp 2.6 = (69.6,\ 74.8)$$

| Parameter | Mean Effect | Std. Dev. | Confidence Interval |
|---|---|---|---|
| $\mu$ | 72.2 | 1.4 | ( 69.6, 74.8) |
| Caches | | | |
| Two Caches | -21.2 | 2.0 | ( -24.9, -17.5) |
| One Cache | -20.2 | 2.0 | ( -23.9, -16.5) |
| No Cache | 41.4 | 2.0 | ( 37.7, 45.1) |

- All three cache alternatives are significantly different from the average.

# Example 21.5 (Cont)

| Parameter | Mean Effect | Std. Dev. | Confidence Interval |
|---|---|---|---|
| ASM | -0.5 | 2.8 | ( -5.8, 4.7)† |
| TECO | 8.8 | 2.8 | ( 3.6, 14.0) |
| SIEVE | -3.5 | 2.8 | ( -8.8, 1.7)† |
| DHRYSTONE | -1.5 | 2.8 | ( -6.8, 3.7)† |
| SORT | -3.2 | 2.8 | ( -8.4, 2.0)† |

† $\Rightarrow$ Not significant

- All workloads, except TECO, are similar to the average and hence to each other.

# Example 21.5: CI for Differences

| | Two Caches | One Cache | No Cache |
|---|---|---|---|
| Two Caches | | ( -7.4, 5.4)† | ( -69.0, -56.2) |
| One Cache | | | ( -68.0, -55.2) |

† ⇒ Not significant

- Two-cache and one-cache alternatives are both significantly better than a no cache alternative.
- There is no significant difference between two-cache and one-cache alternatives.

# Multiplicative Models

- Additive model:

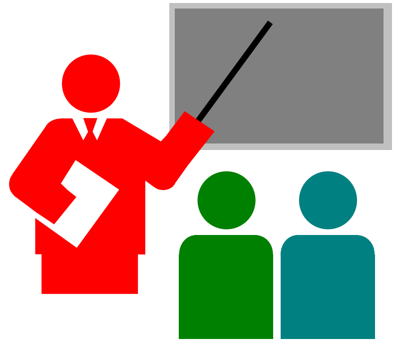
  $$y_i = \mu + \alpha_j + \beta_i + e_{ij}$$

- If factors multiply $\Rightarrow$ Use multiplicative model
- Example: processors and workloads
  - Log of response follows an additive model
- If the spread in the residuals increases with the mean response
  $\Rightarrow$ Use transformation

# Missing Observations

- Recommended Method:
  - Divide the sums by respective number of observations
  - Adjust the degrees of freedoms of sums of squares
  - Adjust formulas for standard deviations of effects
- Other Alternatives:
  - Replace the missing value by $\hat{y}$ such that the residual for the missing experiment is zero.
  - Use y such that SSE is minimum.

# Summary



Two Factor Designs Without Replications

- Model:

$$y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}$$

- Effects are computed so that:

$$\sum_{j=1}^{a} \alpha_j = 0$$
$$\sum_{i=1}^{b} \beta_i = 0$$

- Effects:

$$\mu = \bar{y}_{..};\ \alpha_j = \bar{y}_{.j} - \bar{y}_{..};\ \beta_i = \bar{y}_{i.} - \bar{y}_{..}$$

# Summary (Cont)

- Allocation of variation: SSE can be calculated after computing

$$\begin{array}{ccccccccc} \sum_{ij} y_{ij}^2 & = & ab\mu^2 & + & b\sum_j \alpha_j^2 & + & a\sum_i \beta_i^2 & + & \sum_{ijk} e_{ijk}^2 \\ \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \end{array}$$

Degrees of freedom:

$$\begin{array}{ccccccccc} \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \\ ab & = & 1 & + & (a-1) & + & (b-1) & + & (a-1)(b-1) \end{array}$$

- Mean squares:

$$\text{MSA} = \frac{\text{SSA}}{a-1};\ \text{MSB} = \frac{\text{SSB}}{b-1};\ \text{MSE} = \frac{\text{SSE}}{(a-1)(b-1)}$$

- Analysis of variance:

$\text{MSA}/\text{MSE}$ should be greater than $F_{[1-\alpha;a-1,(a-1)(b-1)]}$.

$\text{MSB}/\text{MSE}$ should be greater than $F_{[1-\alpha;b-1,(a-1)(b-1)]}$.

# Summary (Cont)

- Standard deviation of effects:
  $s_\mu^2 = s_e^2/ab;\ s_{\alpha_j}^2 = s_e^2(a-1)/ab;\ s_{\beta_i}^2 = s_e^2(b-1)/ab;$
- Contrasts:
  For $\sum_{j=1}^a h_j\ \alpha_j,\ \sum_{j=1}^a h_j = 0$: Mean $= \sum_{j=1}^a h_j\ \bar{y}_{.j}$; Variance $= s_e^2 \sum_{j=1}^a h_j^2/b$
  For $\sum_{i=1}^b h_i\ \beta_i,\ \sum_{i=1}^b h_i = 0$: Mean $= \sum_{i=1}^b h_i\ \bar{y}_{i.}$; Variance $= s_e^2 \sum_{i=1}^b h_i^2/a$
- All confidence intervals are calculated using $t_{[1-\alpha/2;(a-1)(b-1)]}$.
- Model assumptions:
  - Errors are IID normal variates with zero mean.
  - Errors have the same variance for all factor levels.
  - The effects of various factors and errors are additive.
- Visual tests:
  - No trend in scatter plot of errors versus predicted responses
  - The normal quantile-quantile plot of errors should be linear.

# Homework 21: Exercise 21.1

Execution Times

| Workloads | Processors | | |
|---|---|---|---|
| | Scheme86 | Spectrum125 | Spectrum62.5 |
| Garbage Collection | 39.97 | 99.06 | 56.24 |
| Pattern Match | 0.958 | 1.672 | 1.252 |
| Bignum Addition | 0.01910 | 0.03175 | 0.01844 |
| Bignum Multiplication | 0.256 | 0.423 | 0.236 |
| Fast Fourier Transform (1024) | 10.21 | 20.28 | 10.14 |

Analyze the data of Case study 21.2 using a 2-factor additive model.

- Estimate effects and prepare ANOVA table
- Plot residuals as a function of predicted response.
- Also, plot a normal quantile-quantile plot for the residuals.
- Determine 90% confidence intervals for the paired differences. (Confidence intervals of $\alpha_1$-$\alpha_2$, $\alpha_1-\alpha_3$, $\alpha_2$-$\alpha_3$)
- Are the processors significantly different?
- Discuss what indicators in the data, analysis, or plot would suggest that this is not a good model.