Introduction to Time Series Analysis

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Audio/Video recordings of this lecture are available at:

http://www.cse.wustl.edu/~jain/cse567-15/

- What is a time series?
- Autoregressive Models
- Moving Average Models
- Integrated Models
- ARMA, ARIMA, SARIMA, FARIMA models
- Note: These slides are based on R. Jain, "The Art of Computer Systems Performance Analysis," 2nd Edition (in preparation).

Stochastic Processes

- Ordered sequence of random observations
- Example:
  - Number of virtual machines in a server
  - Number of page faults
  - Number of queries over time
- Analysis Technique: Time Series Analysis
- Long-range dependence and self-similarity in such processes can invalidate many previous results

Stochastic Processes: Key Questions

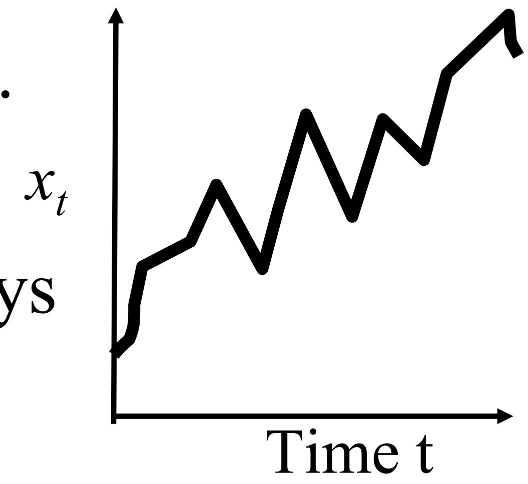
1. What is a time series?
2. What are different types of time series models?
3. How to fit a model to a series of measured data?
4. What is a stationary time series?
5. Is it possible to model a series that is not stationary?
6. How to model a series that has a periodic or seasonal behavior as is common in video streaming?

Stochastic Processes : Key Questions (Cont)

1. What are heavy-tailed distributions and why they are important?
2. How to check if a sample of observations has a heavy tail?
3. What are self-similar processes?
4. What are short-range and long-range dependent processes?
5. Why long-range dependence invalidates many conclusions based on previous statistical methods?
6. How to check if a sample has a long-range dependence?

What is a Time Series

- Time series = Stochastic Process
- A sequence of observations over time.
- Examples:
  - Price of a stock over successive days
  - Sizes of video frames
  - Sizes of packets over network
  - Sizes of queries to a database system
  - Number of active virtual machines in a cloud
- Goal: Develop models of such series for resource allocation and improving user experience.



Autoregressive Models

- Predict the variable as a linear regression of the immediate past value: $\hat{x}_t = a_0 + a_1 x_{t-1}$
- Here, $\hat{x}_t$ is the best estimate of x_t given the past history $\{x_0, x_1, \ldots, x_{t-1}\}$
- Even though we know the complete past history, we assume that x_t can be predicted based on just x_{t-1}.
- Auto-Regressive = Regression on Self
- Error: $e_t = x_t - \hat{x}_t = x_t - a_0 - a_1 x_{t-1}$
- Model: $x_t = a_0 + a_1 x_{t-1} + e_t$
- Best a_0 and a_1 $\Rightarrow$ minimize the sum of square of errors

Example 37.1

- The number of disk access for 50 database queries were measured to be: 73, 67, 83, 53, 78, 88, 57, 1, 29, 14, 80, 77, 19, 14, 41, 55, 74, 98, 84, 88, 78, 15, 66, 99, 80, 75, 124, 103, 57, 49, 70, 112, 107, 123, 79, 92, 89, 116, 71, 68, 59, 84, 39, 33, 71, 83, 77, 37, 27, 30.
- For this data: $\sum_{t=2}^{50} x_t = 3313 \quad \sum_{t=2}^{50} x_{t-1} = 3356$

$$\sum_{t=2}^{50} x_t x_{t-1} = 248147 \quad \sum_{t=2}^{50} x_{t-1}^2 = 272102 \quad n = 49$$

$$\begin{aligned} a_0 &= \frac{\sum x_t \sum x_{t-1}^2 - \sum x_{t-1} \sum x_t x_{t-1}}{n \sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2} \\ &= \frac{3313 \times 272102 - 3356 \times 248147}{49 \times 272102 - 3356^2} = 33.181 \end{aligned}$$

Example 37.1 (Cont)

$$a_1 = \frac{n\sum x_t x_{t-1} - \sum x_t \sum x_{t-1}}{n\sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2}$$

$$= \frac{49 \times 248147 - 3313 \times 3356}{49 \times 272102 - 3356^2} = 0.503$$

- The AR(1) model for the series is:

$$x_t = 33.181 + 0.503 x_{t-1} + e_t$$

- The predicted value of x_2 given x_1 is:

$$\hat{x}_2 = a_0 + a_1 x_1 = 33.181 + 0.503 \times 73 = 69.880$$

- The actual observed value of is 67. Therefore, the prediction error is:

$$e_2 = x_2 - \hat{x}_2 = 67 - 69.880 = -2.880$$

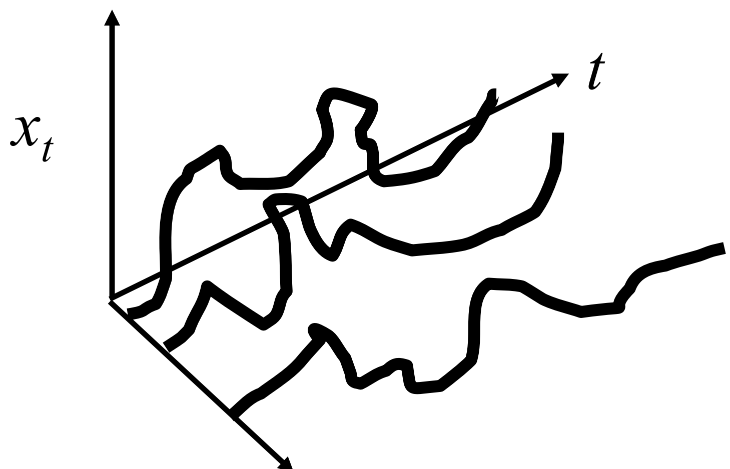
- Sum of squared errors SSE = 32995.57

Exercise 37.1

- Fit an AR(1) model to the following sample of 50 observations: 83, 86, 46, 34, 130, 109, 100, 81, 84, 148, 93, 76, 69, 40, 50, 56, 63, 104, 35, 55, 124, 52, 55, 81, 33, 76, 83, 90, 94, 37, -2, 33, 105, 133, 78, 50, 115, 149, 98, 110, 25, 82, 59, 80, 43, 58, 88, 78, 55, 68. Find a_0, a_1 and the minimum SSE.

Stationary Process

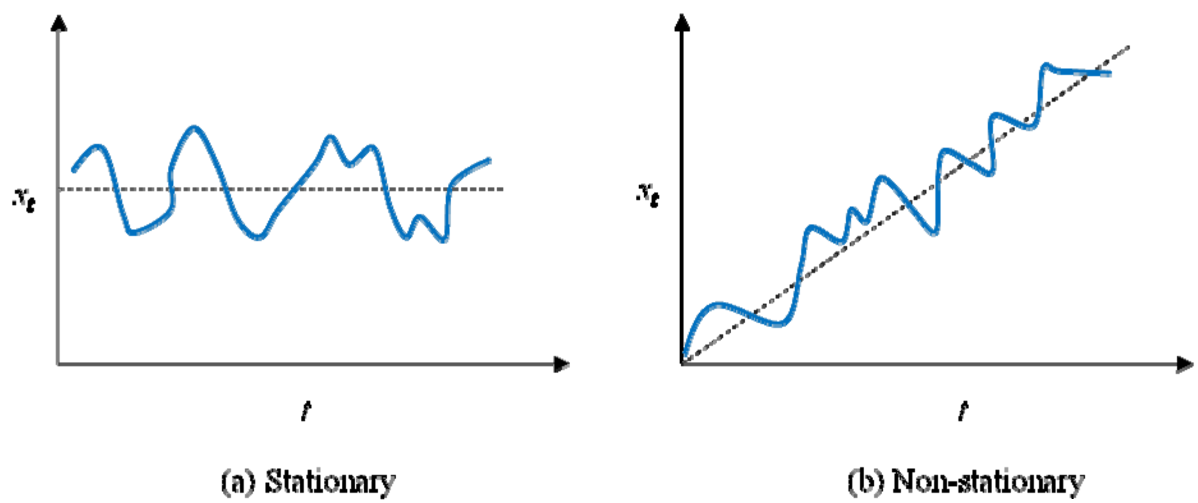
- Each realization of a random process will be different:



- x is function of the realization i (space) and time t: $x(i, t)$
- We can study the distribution of x_t in space.
- Each x_t has a distribution, e.g., Normal $f(x_t) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_t-\mu)^2}{2\sigma^2}}$
- If this same distribution (normal) with the same parameters μ, σ applies to x_{t+1}, x_{t+2}, …, we say x_t is stationary.

Stationary Process (Cont)

- Stationary = Standing in time
  ⇒ Distribution does not change with time.
- Similarly, the joint distribution of x_t and x_{t-k} depends only on k not on t.
- The joint distribution of x_t, x_{t-1}, …, x_{t-k} depends only on k not on t.



Autocorrelation

- Covariance of x_t and x_{t-k} = Auto-covariance at lag k

  $$\text{Autocovariance of } x_t \text{ at lag } k = \text{Cov}[x_t, x_{t-k}] = E[(x_t - \mu)(x_{t-k} - \mu)]$$

- For a stationary series:
  - Statistical characteristics do not depend upon time t.
  - Autocovariance depends only on lag k and not on time t

$$\begin{aligned}\text{Autocorrelation of } x_t \text{ at lag } k \;\; r_k &= \frac{\text{Autocovariance of } x_t \text{ at lag } k}{\text{Variance of } x_t} \\ &= \frac{\text{Cov}[x_t, x_{t-k}]}{\text{Var}[x_t]} \\ &= \frac{E[(x_t - \mu)(x_{t-k} - \mu)]}{E[(x_t - \mu)^2]}\end{aligned}$$

- Autocorrelation is dimensionless and is easier to interpret than autocovariance.

Example 37.2

- For the data of Example 37.1, the variance and covariance's at lag 1 and 2 are computed as follows:

$$\text{Sample Mean } \bar{x} = \frac{1}{50}\sum_{t=1}^{50} x_t = \frac{3386}{50} = 67.72$$

$$Var(x_t) = E[(x_t - \mu)^2] = \frac{1}{49}\sum_{t=1}^{50}(x_t - \bar{x})^2 = \frac{273002 - 50 \times 67.72^2}{49} = 891.879$$

Example 37.2 (Cont)

$$Cov(x_t, x_{t-1}) = E[(x_t - \mu)(x_{t-1} - \mu)]$$

$$= \frac{1}{49}\sum_{t=2}^{50}(x_t - \bar{x}_t)(x_{t-1} - \bar{x}_{t-1})$$

$$= \frac{1}{49}\left[\sum_{t=2}^{50} x_t x_{t-1} - \left(\frac{1}{49}\sum_{t=2}^{50} x_t\right)\sum_{t=2}^{50} x_{t-1} - \sum_{t=2}^{50} x_t\left(\frac{1}{49}\sum_{t=2}^{50} x_{t-1}\right)\right.$$

$$\left. + 49\left(\frac{1}{49}\sum_{t=2}^{50} x_t\right)\left(\frac{1}{49}\sum_{t=2}^{50} x_{t-1}\right)\right]$$

$$= \frac{1}{49}\left[\sum_{t=2}^{50} x_t x_{t-1} - \frac{1}{49}\left(\sum_{t=2}^{50} x_t\right)\left(\sum_{t=2}^{50} x_{t-1}\right)\right]$$

$$= \frac{1}{49}\left[248147 - \frac{3313 \times 3356}{49}\right] = 433.476$$

- Small Sample $\Rightarrow \bar{x}_t$ and $\bar{x}_{t-1}$ are slightly different. Not so for large samples.

Example 37.2 (Cont)

$$
\begin{aligned}
Cov(x_t, x_{t-2}) &= E[(x_t - \mu)(x_{t-2} - \mu)] \\
&= \frac{1}{48}\sum_{t=3}^{50}(x_t - \overline{x}_t)(x_{t-2} - \overline{x}_{t-2}) \\
&= \frac{1}{48}\left[\sum_{t=3}^{50} x_t x_{t-2} - \frac{1}{48}\left(\sum_{t=3}^{50} x_t\right)\left(\sum_{t=3}^{50} x_{t-2}\right)\right] \\
&= \frac{1}{48}\left[229360 - \frac{3246 \times 3329}{48}\right] \\
&= 88.258
\end{aligned}
$$

- Note: Only 48 pairs of $\{x_t, x_{t-1}\}$ ⇒ Divisor is 48

Example 37.2 (Cont)

$$Autocorrelation\ at\ lag\ 0 = r_0 = \frac{Var(x_t)}{Var(x_t)} = \frac{891.879}{891.879} = 1$$

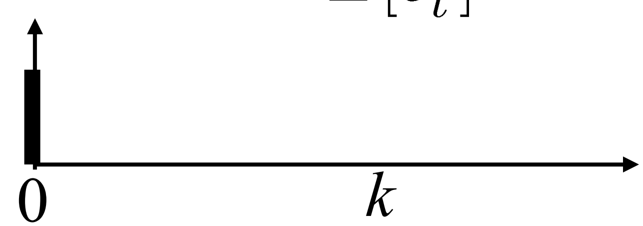
$$Autocorrelation\ at\ lag\ 1 = r_1 = \frac{Cov(x_t, x_{t-1})}{Var(x_t)} = \frac{433.476}{891.879} = 0.486$$

$$Autocorrelation\ at\ lag\ 1 = r_1 = \frac{Cov(x_t, x_{t-1})}{Var(x_t)} = \frac{433.476}{891.879} = 0.486$$

White Noise

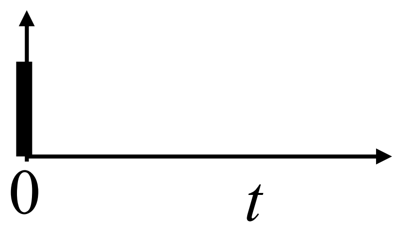
- Errors e_t are normal independent and identically distributed (IID) with zero mean and variance σ^2
- Such IID sequences are called "**white noise**" sequences.
- Properties:

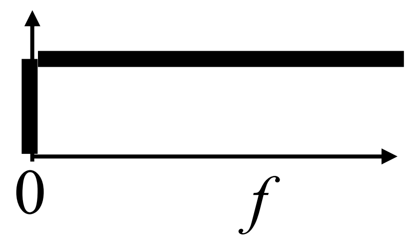
$$
\begin{aligned}
E[e_t] &= 0 \qquad \forall t \\
\mathrm{Var}[e_t] &= E[e_t^2] = \sigma^2 \qquad \forall t \\
\mathrm{Cov}[e_t, e_{t-k}] &= E[e_t e_{t-k}] = \begin{cases} \sigma^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \\
\mathrm{Cor}[e_t, e_{t-k}] &= \frac{E[e_t e_{t-k}]}{E[e_t^2]} = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}
\end{aligned}
$$



White Noise (Cont)

- The autocorrelation function of a white noise sequence is a spike (δ function) at k=0.
- The Laplace transform of a δ function is a constant. So in frequency domain white noise has a flat frequency spectrum.





- It was incorrectly assumed that white light has no color and, therefore, has a flat frequency spectrum and so random noise with flat frequency spectrum was called white noise.

Ref: http://en.wikipedia.org/wiki/Colors_of_noise

White Noise Autocorrelations

- It can be shown that autocorrelations for white noise are normally distributed with mean:

$$E[r_k] \approx \frac{-1}{n}$$

  and variance:

$$\mathrm{Var}[r_k] \approx \frac{1}{n}$$

- Therefore, their 95% confidence interval is $-1/n \mp 1.96/\sqrt{n}$

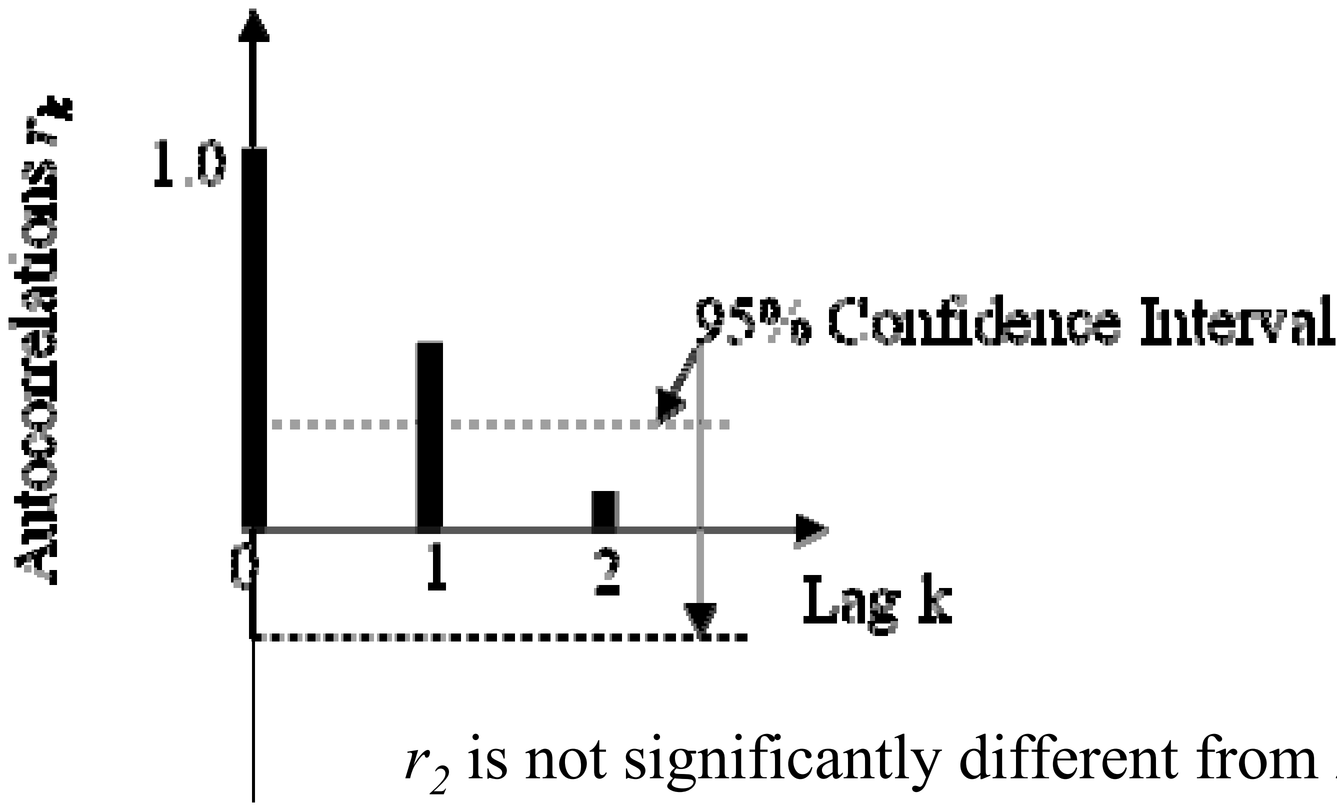
  This is generally approximated as $\mp 2/\sqrt{n}$

- This confidence interval can be used to check if a particular autocorrelation is zero.

Example 37.3

- For the data of Example 37.1: n=50

$$CI = \mp 2/\sqrt(50) = \mp 0.283$$



r_2 is not significantly different from zero.

Exercise 37.2

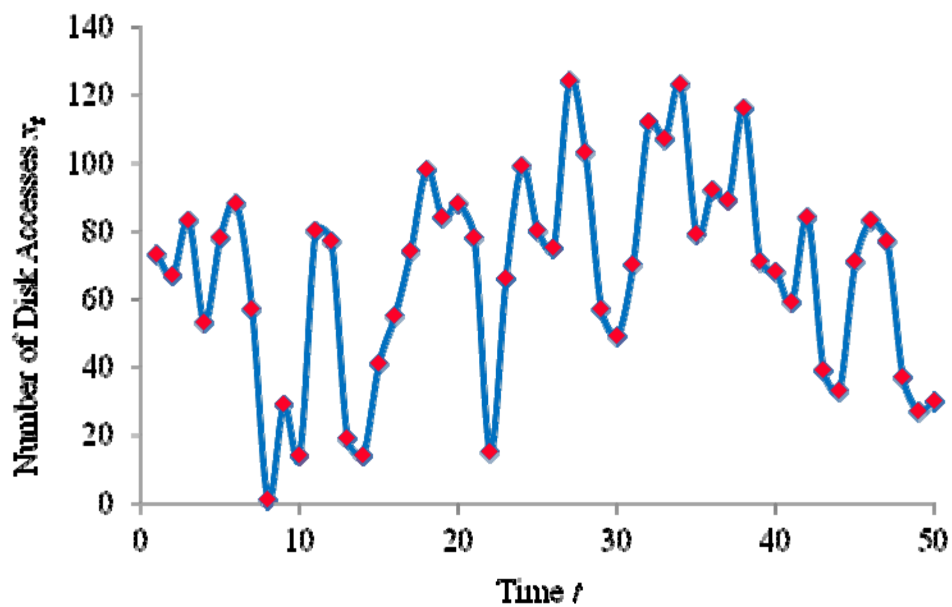
- Determine autocorrelations at lag 0 through 2 for the data of Exercise 37.1 and determine which of these autocorrelations are significant at 95% confidence.

Assumptions for AR(1) Models

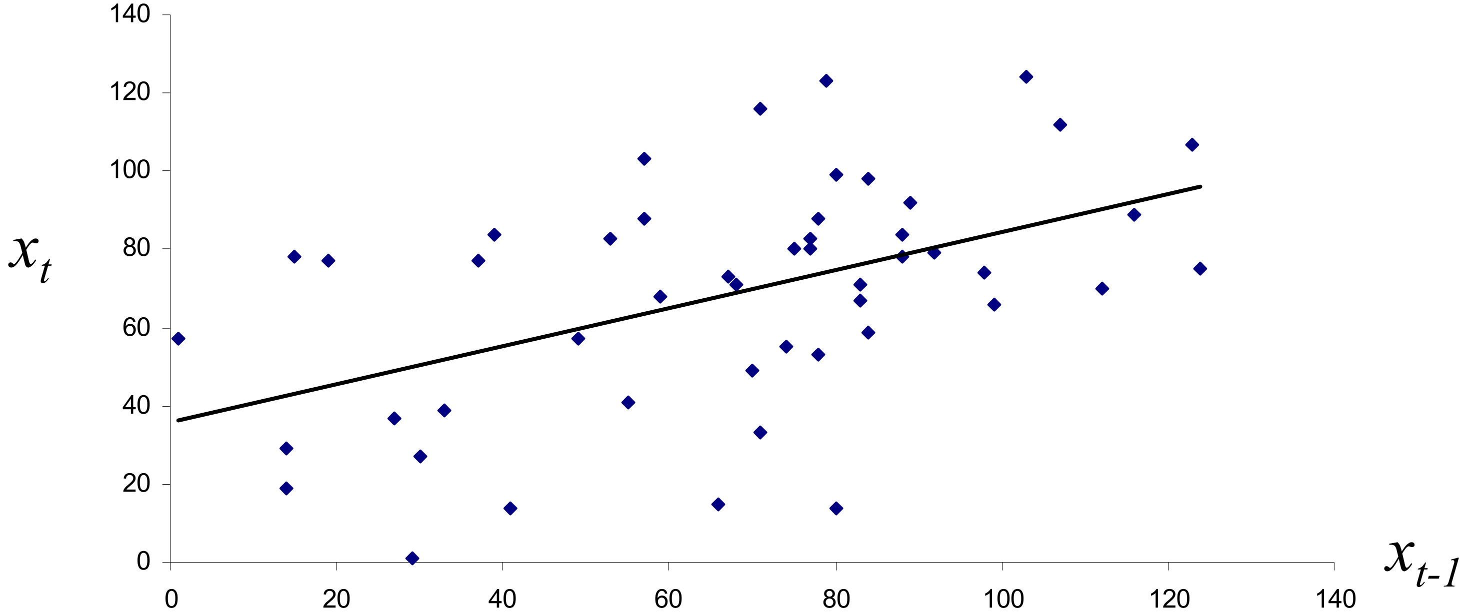
- x_t is a Stationary process
- Linear relationship between successive values
- Normal Independent identically distributed errors:
  - Normal errors
  - Independent errors
- Additive errors

Visual Tests for AR(1) Models

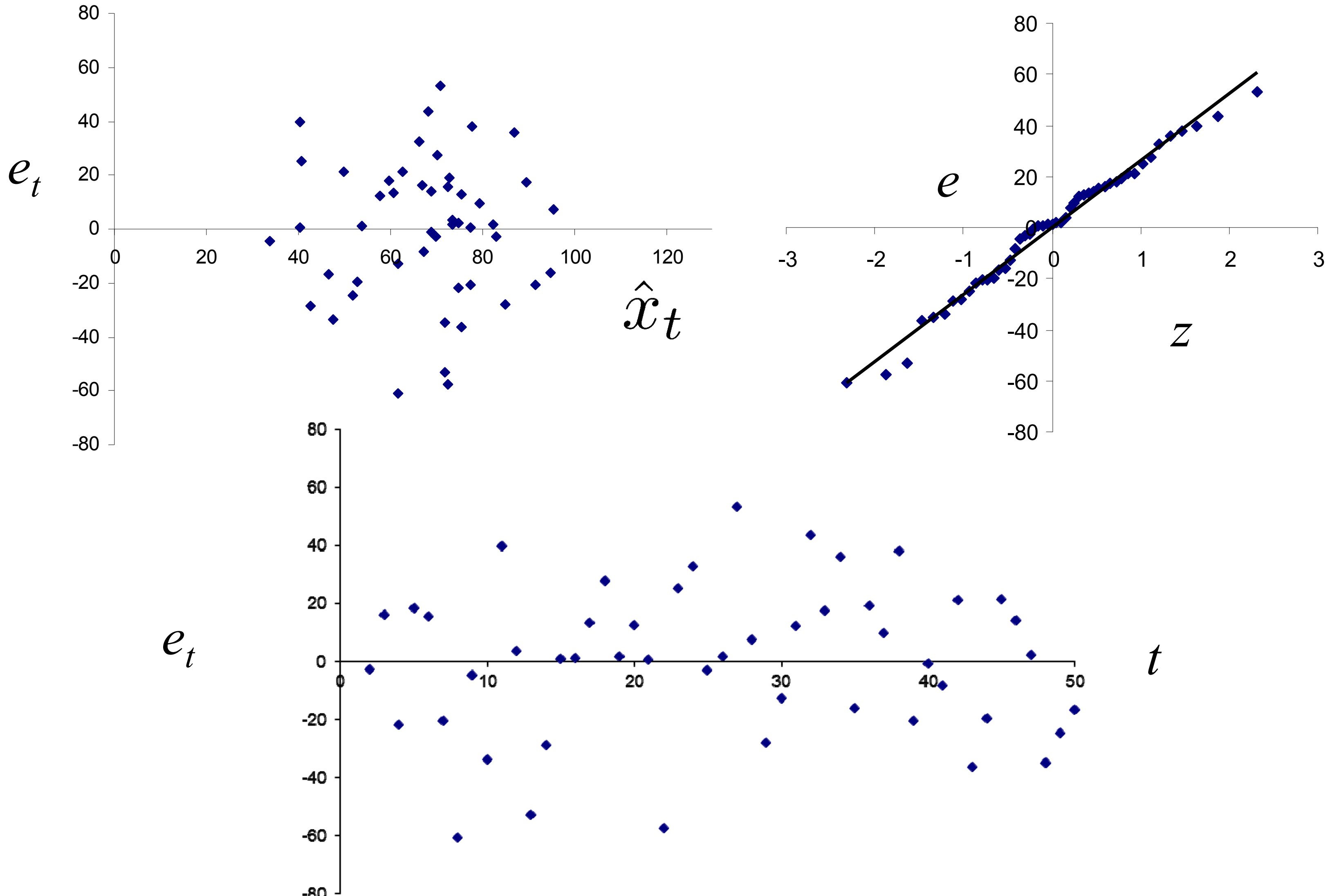
1. Plot x_t as a function of t and look for trends
2. x_t vs. x_{t-1} for linearity
3. Errors e_t vs. predicted values $\hat{x}_t$ for additivity
4. Q-Q Plot of errors for Normality
5. Errors e_t vs. t for iid



Visual Tests (Cont)



Visual Tests (Cont)



Exercise 37.3

- Conduct visual tests to verify whether or not the AR(1) model fitted in Exercise 37.1 is appropriate .

AR(p) Model

- x_t is a function of the last p values:

$$x_t = a_0 + a_1 x_{t-1} + a_2 x_{t-2} + \cdots + a_p x_{t-p} + e_t$$

- AR(2): $x_t = a_0 + a_1 x_{t-1} + a_2 x_{t-2} + e_t$

- AR(3): $x_t = a_0 + a_1 x_{t-1} + a_2 x_{t-2} + a_3 x_{t-3} + e_t$

Backward Shift Operator

$$B(x_t) = x_{t-1}$$

- Similarly, $B(B(x_t)) = B(x_{t-1}) = x_{t-2}$
- Or

$$B^2 x_t = x_{t-2}$$
$$B^3 x_t = x_{t-3}$$
$$B^k x_t = x_{t-k}$$

- Using this notation, AR(p) model is:

$$\begin{aligned} x_t - a_1 x_{t-1} - a_2 x_{t-2} - \cdots - a_p x_{t-p} &= a_0 + e_t \\ x_t - a_1 B x_t - a_2 B^2 x_t - \cdots - a_p B^p x_t &= a_0 + e_t \\ (1 - a_1 B - a_2 B^2 - \cdots - a_p B^p) x_t &= a_0 + e_t \\ \phi_p(B) x_t &= a_0 + e_t \end{aligned}$$

- Here, ϕ_p is a polynomial of degree p.

AR(p) Parameter Estimation

$$x_t = a_0 + a_1 x_{t-1} + a_2 x_{t-2} + e_t$$

- The coefficients a_i's can be estimated by minimizing SSE using Multiple Linear Regression.

$$\text{SSE} = \sum e_t^2 = \sum_{t=3}^{n} (x_t - a_0 - a_1 x_{t-1} - a_2 x_{t-2})^2$$

- Optimal a_0, a_1, and a_2 $\Rightarrow$ Minimize SSE
$\Rightarrow$Set the first differential to zero:

$$\frac{d}{da_0}\text{SSE} = \sum_{t=3}^{n} -2(x_t - a_0 - a_1 x_{t-1} - a_2 x_{t-2}) = 0$$

$$\frac{d}{da_1}\text{SSE} = \sum_{t=3}^{n} -2x_{t-1}(x_t - a_0 - a_1 x_{t-1} - a_2 x_{t-2}) = 0$$

$$\frac{d}{da_2}\text{SSE} = \sum_{t=3}^{n} -2x_{t-2}(x_t - a_0 - a_1 x_{t-1} - a_2 x_{t-2}) = 0$$

AR(p) Parameter Estimation (Cont)

- The equations can be written as:

$$\begin{bmatrix} n-2 & \sum x_{t-1} & \sum x_{t-2} \\ \sum x_{t-1} & \sum x_{t-1}^2 & \sum x_{t-1}x_{t-2} \\ \sum x_{t-2} & \sum x_{t-1}x_{t-2} & \sum x_{t-2}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum x_t \\ \sum x_t x_{t-1} \\ \sum x_t x_{t-2} \end{bmatrix}$$

  Note: All sums are for t=3 to n. n-2 terms.

- Multiplying by the inverse of the first matrix, we get:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} n-2 & \sum x_{t-1} & \sum x_{t-2} \\ \sum x_{t-1} & \sum x_{t-1}^2 & \sum x_{t-1}x_{t-2} \\ \sum x_{t-2} & \sum x_{t-1}x_{t-2} & \sum x_{t-2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum x_t \\ \sum x_t x_{t-1} \\ \sum x_t x_{t-2} \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} n-p & \sum x_{t-1} & \sum x_{t-2} & \cdots & \sum x_{t-p} \\ \sum x_{t-1} & \sum x_{t-1}^2 & \sum x_{t-1}x_{t-2} & \cdots & \sum x_{t-1}x_{t-p} \\ \sum x_{t-2} & \sum x_{t-1}x_{t-2} & \sum x_{t-2}^2 & \cdots & \sum x_{t-2}x_{t-p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum x_{t-p} & \sum x_{t-1}x_{t-p} & \sum x_{t-2}x_{t-p} & \cdots & \sum x_{t-p}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum x_t \\ \sum x_t x_{t-1} \\ \sum x_t x_{t-2} \\ \vdots \\ \sum x_t x_{t-p} \end{bmatrix}$$

- All sums are from t=p to t=n and have n-p terms.
- For larger data sets: r_k is the autocorrelation at lag k

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} 1 & r_1 & \cdots & r_{p-1} \\ r_1 & r_2 & \cdots & r_p \\ \vdots & \vdots & \vdots & \vdots \\ r_{p-1} & r_{p-2} & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{bmatrix}$$

$$a_0 = (1 - a_1 - a_2 - \cdots - a_p)\overline{x}$$

Example 37.5

- Consider the data of Example 37.1 and fit an AR(2) model:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.486 \\ 0.486 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.486 \\ 0.099 \end{bmatrix}$$

$$= \begin{bmatrix} 0.575 \\ -0.182 \end{bmatrix}$$

$$a_0 = (1 - a_1 - a_2)\bar{x} = (1 - 0.575 + 0.182)67.72 = 41.164$$

- SSE= 31979.32
- Small sample ⇒ Values of a0, a1, and a2 are approximate.
- Exact model by regression:

$$x_t = 39.979 + 0.587 x_{t-1} - 0.180 x_{t-2} + e_t$$

SSE=31969.99

Exercise 37.4

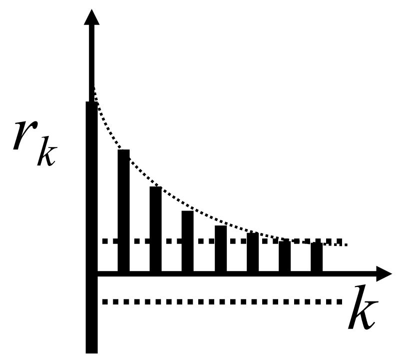
- Fit an AR(2) model to the data of Exercise 37.1. Determine parameters a_0, a_1, a_2 and the SSE using multiple regression. Repeat the determination of parameters using autocorrelation function values.

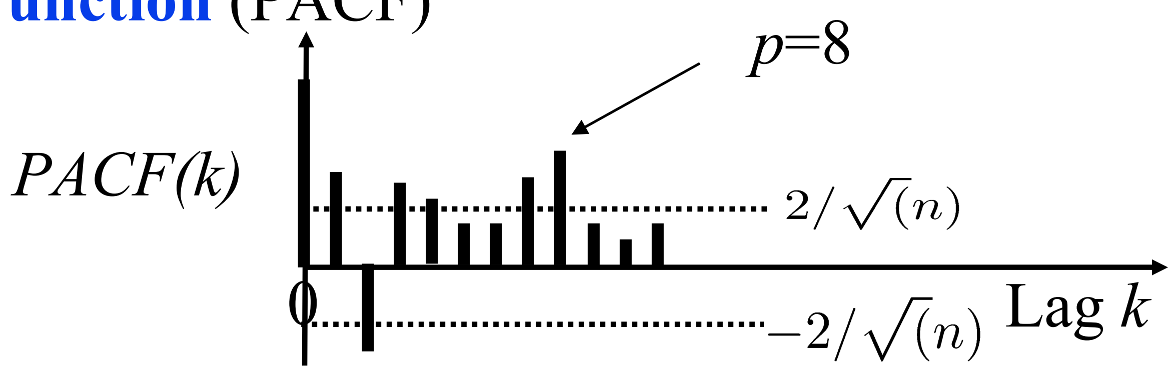
Exercise 37.5

- Fit an AR(3) model to the data of Exercise 37.1. Determine parameters a_0, a_1, a_2, a_3 and the SSE using multiple regression.

Determining the Order AR(p)

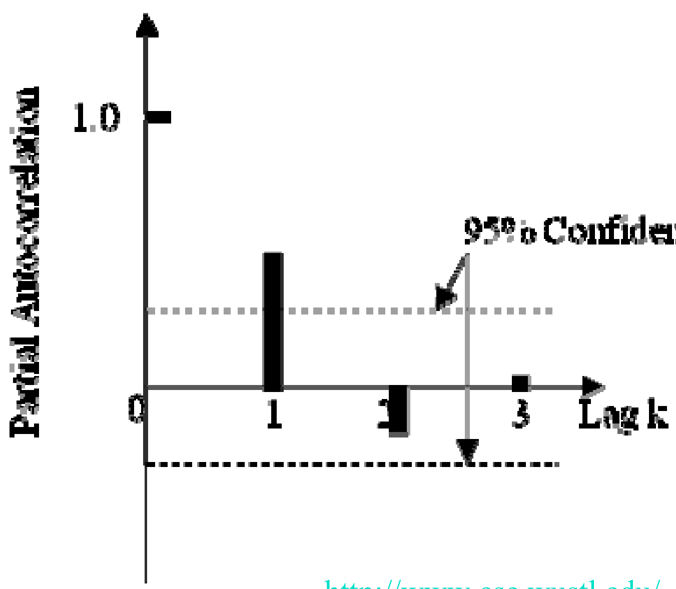
- ACF of AR(1) is an exponentially decreasing fn of k
- Fit AR(p) models of order p=0, 1, 2, …
- Compute the confidence intervals of a_p. $a_p \mp 2/\sqrt(n)$
- After some p, the last coefficients a_p will not be significant for all higher order models.
- This highest p is the order of the AR(p) model for the series.
- This sequence of last coefficients is also called "**Partial Autocorrelation Function** (PACF)"





Example 37.6

- For the data of Example 37.1, we have:
- AR(1): $x_t = 33.181 + 0.503 x_{t-1} + e_t$
- AR(2): $x_t = 39.979 + 0.587 x_{t-1} - 0.180 x_{t-2} + e_t$
- Similarly, AR(3): $x_t = 37.313 + 0.598 x_{t-1} - 0.211 x_{t-2} + 0.052 x_{t-3} + e_t$
- PACF at lags 1, 2, and 3 are: 0.503, -0.180, and 0.052



AR(1) is appropriate.

Computing PACF

PACF at lag 1 = $s_1 = a_1$ in AR(1) $= r_1$

$|M|$ = Determinant of M

PACF at lag 2 = $s_2 = a_2$ in AR(2) $= \dfrac{\begin{vmatrix} 1 & r_1 \\ r_1 & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & r_1 \\ r_1 & 1 \end{vmatrix}}$

PACF at lag 3 = $s_3 = a_3$ in AR(3) $= \dfrac{\begin{vmatrix} 1 & r_1 & r_1 \\ r_1 & 1 & r_2 \\ r_2 & r_1 & r_3 \end{vmatrix}}{\begin{vmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{vmatrix}}$

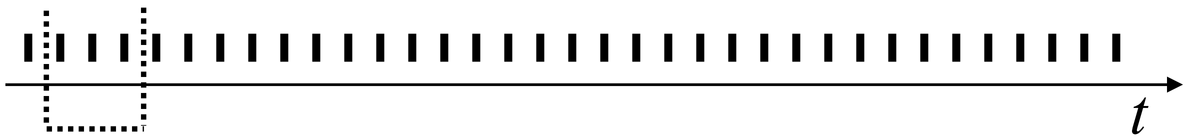
Computing PACF (Cont)

$$\text{PACF at lag } k = s_k = a_k \text{ in AR}(k) = \frac{\begin{vmatrix} 1 & r_1 & \cdots & r_1 \\ r_1 & 1 & \cdots & r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1} & r_{k-2} & \cdots & r_k \end{vmatrix}}{\begin{vmatrix} 1 & r_1 & \cdots & r_{k-1} \\ r_1 & 1 & \cdots & r_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1} & r_{k-2} & \cdots & 1 \end{vmatrix}}$$

Exercise 37.6

- Using the results of Exercises 37.1, 37.4, and 37.5, determine the partial autocorrelation function at lags 1, 2, 3 for the data of Exercise 37.1. Determine which values are significant. Based on this which AR(p) model will be appropriate for this data?

Moving Average (MA) Models



- Moving Average of order 1: MA(1)

  $$x_t - b_0 = e_t + b_1 e_{t-1}$$

  b_0 is the mean of the time series.
- The parameters b_0 and b_1 cannot be estimated using standard regression formulas since we do not know errors. The errors depend on the parameters.
- So the only way to find optimal b_0 and b_1 is by iteration.
  $\Rightarrow$ Start with some suitable values and change b_0 and b_1 until SSE is minimized and average of errors is zero.

Example 37.4

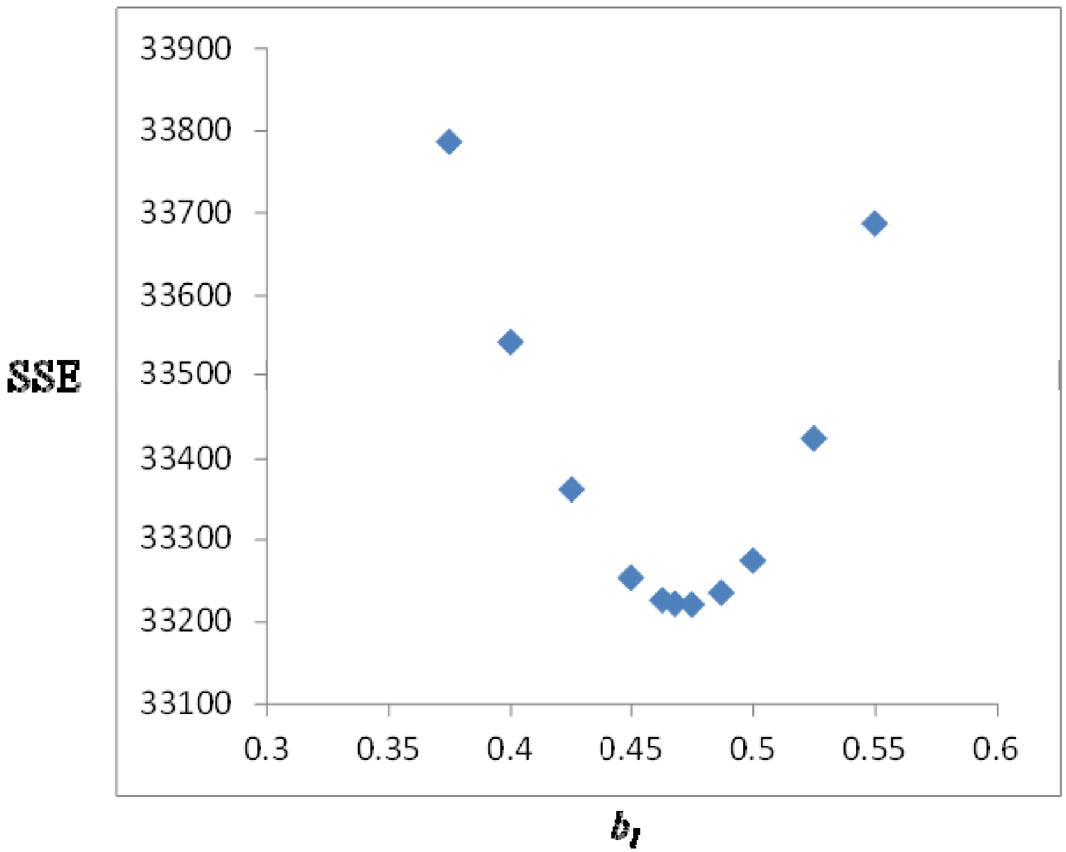
- Consider the data of Example 37.1.
- For this data: $\bar{x} = \frac{1}{50}\sum_{t=1}^{50} x_t = 67.72$
- We start with b_0 = 67.72, b_1=0.4,
  Assuming e_0=0, compute all the errors and SSE.

  $$\bar{e} = \frac{1}{50}\sum_{t=1}^{50} e_t = -0.152 \qquad \text{and SSE} = 33542.8$$

- We then adjust a_0 and b_1 until SSE is minimized and mean error is close to zero.

Example 37.4 (Cont)

- The steps are: Starting with $b_0 = \bar{x}$ and trying various values of b_1. SSE is minimum at b_1=0.475. SSE= 33221.06

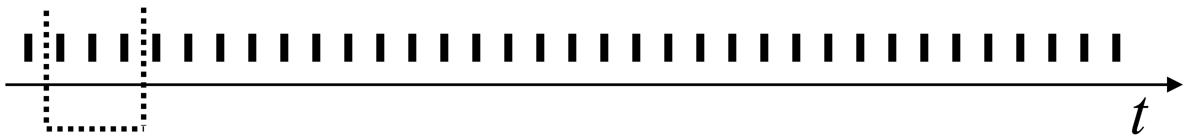


Example 37.4 (Cont)

$$\bar{e} = \frac{1}{50}\sum_{t=1}^{50} e_t = -0.1661$$

- Keeping b_1=0.475, try neighboring values of b_0 to get average error as close to zero as possible.
- b_0= 67.475 gives $\bar{e}$=-0.001 SSE=33221.93

MA(q) Models



- Moving Average of order 1: MA(1)
  $$x_t - b_0 = e_t + b_1 e_{t-1}$$
- Moving Average of order 2: MA(2)
  $$x_t - b_0 = e_t + b_1 e_{t-1} + b_2 e_{t-2}$$
- Moving Average of order q: MA(q)
  $$x_t - b_0 = e_t + b_1 e_{t-1} + b_2 e_{t-2} + \cdots + b_q e_{t-q}$$
- Moving Average of order 0: MA(0) (Note: This is also AR(0))
  $$x_t - b_0 = e_t$$
  x_t-b_0 is a white noise. b_0 is the mean of the time series.

Exercise 37.7

- Fit an MA(0) model to the data of Exercise 37.1. Determine parameter b_0 and SSE

MA(q) Models (Cont)

- Using the backward shift operator B, MA(q):

$$\begin{aligned} x_t - b_0 &= e_t + b_1 B e_t + b_2 B^2 e_t + \cdots + b_q B^q e_t \\ &= (1 + b_1 B + b_2 B^2 + \cdots + b_q B^q) e_t \\ &= \Psi_q(B) e_t \end{aligned}$$

- Here, Ψ_q is a polynomial of order q.

Example 37.8

- Fit MA(2) model to the data of Example 37.1

$$x_t = b_0 + e_t + b_1 e_{t-1} + b_2 e_{t-2}$$

- Round 1: Setting $b_0 = \bar{x}_t = 67.72$ we try 9 combinations of b_1={0.2,0.3,0.4} and b_2={0.2, 0.3, 0.4}.
  Minimum SSE is 33490.26 at b_1=0.4 and b_2=0.2
- Round 2: Try 4 new points around the current minimum b_0={0.35, 0.45} and b_2={0.15, 0.25}
  Minimum SSE is 32551.62 at b_1=0.45, b_2=0.15
- Round 3: Try 4 new points around the current minimum. Try b_1={0.425, 0.475} and b_2={0.125, 0.175}
  Minimum SSE is 32342.61 at b_1=0.475, b_2=0.125

Example 37.8 (Cont)

- Round 4: Try 4 new points around the current minimum.
  Try b_1={0.4625, 0.4875} and b_2={0.125, 0.175}
  Minimum SSE is 32201.58 at b_1=0.4875, b_2=0.125
- Round 5: Try 4 new points around the current minimum.
  Try b_1={0.481, 0.493} and b_2={0.112, 0.137}
  Minimum SSE is 32148.21 at b_1=0.493, b_2=0.137
- Since the decrease in SSN is small (close to 0.1%), we arbitrarily stop here.
- The model is:

$$x_t = 67.72 + e_t + 0.493e_{t-1} + 0.137e_{t-2}$$

Exercise 38.8

- Fit an MA(1) model to the data of Exercise 37.1. Determine parameters b_0, b_1 and the minimum SSE.

Autocorrelations for MA(1)

- For this series, the mean is:

$$\mu = E[x_t] = a_0 + E[e_t] + b_1 E[e_{t-1}] = a_0$$

- The variance is:

$$\begin{aligned}\text{Var}[x_t] &= E[(x_t - \mu)^2] = E[(e_t + b_1 e_{t-1})^2] \\ &= E[e_t^2 + 2b_1 e_t e_{t-1} + b_1^2 e_{t-1}^2] \\ &= E[e_t^2] + 2b_1 E[e_t e_{t-1}] + b_1^2 E[e_{t-1}^2] \\ &= \sigma^2 + 2b_1 \times 0 + b_1^2 \sigma^2 = (1 + b_1^2)\sigma^2\end{aligned}$$

- The autocovariance at lag 1 is:

$$\begin{aligned}\text{autocovar at lag 1} &= E[(x_t - \mu)(x_{t-1} - \mu)] \\ &= E[e_t + b_1 e_{t-1})(e_{t-1} + b_1 e_{t-2})] \\ &= E[e_t e_{t-1} + b_1 e_{t-1} e_{t-1} + b_1 e_t e_{t-2} + b_1^2 e_{t-1} e_{t-2}] \\ &= E[0 + b_1 E[e_{t-1}^2] + 0 + 0] \\ &= b_1 \sigma^2\end{aligned}$$

Autocorrelations for MA(1) (Cont)

- The autocovariance at lag 2 is:

$$\begin{aligned}\text{Covar at lag 2} &= E[(x_t-\mu)(x_{t-2}-\mu)] \\ &= E[(e_t+b_1e_{t-1})(e_{t-2}+b_1e_{t-3})] \\ &= E[e_te_{t-2}+b_1e_{t-1}e_{t-2}+b_1e_te_{t-3}+b_1^2e_{t-1}e_{t-3}] \\ &= 0+0+0+0=0\end{aligned}$$

- For MA(1), the autocovariance at all higher lags (k>1) is 0.
- The autocorrelation is:

$$r_k = \begin{cases} 1 & k=0 \\ \frac{b_1}{1+b_1^2} & k=1 \\ 0 & k>1 \end{cases}$$

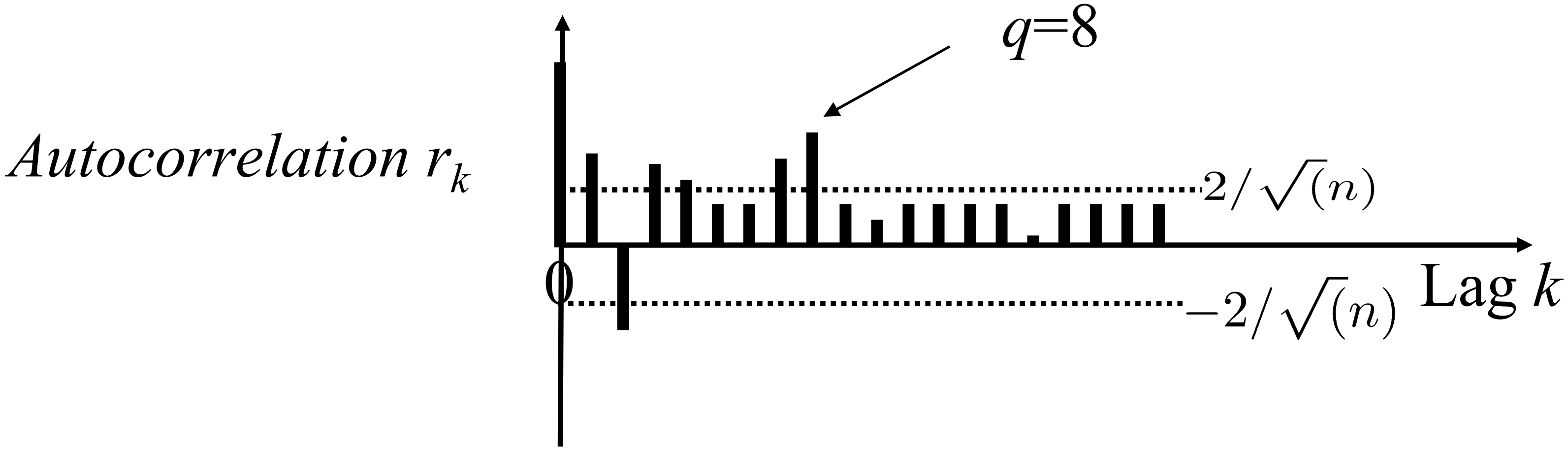
- The autocorrelation of MA(q) series is non-zero only for lags $k \leq q$ and is zero for all higher lags.

Example 37.9

- For the data of Example 37.1:
- Autocorrelation is zero for all lags k >1.
- MA(1) model is appropriate for this data.

Example 37.10

- The order of the last significant r_k determines the order of the MA(q) model.
- For the following data, all autocorrelations at lag 9 and higher are zero ⇒ MA(8) model would be appropriate



Exercise 37.9

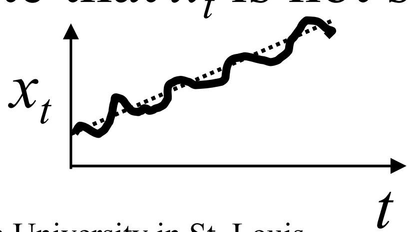
- Fit an MA(2) model to the data of Exercise 37.2. Determine parameters b_0, b_1, b_2 and the minimum SSE. For this data, which model would you choose MA(0), MA(1) or MA(2) and why?

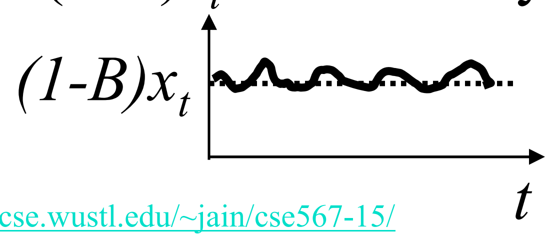
Duality of AR(p) vs. MA(q)

- Determining the coefficients of AR(p) is straight forward but determining the order p requires an iterative procedure
- Determining the order q of MA(q) is straight forward but determining the coefficients requires an iterative procedure

Non-Stationarity: Integrated Models

- In the white noise model AR(0): $x_t = a_0 + e_t$
- The mean a_0 is independent of time.
- If it appears that the time series in increasing approximately linearly with time, the first difference of the series can be modeled as white noise: $(x_t - x_{t-1}) = a_0 + e_t$
- Or using the B operator: $(1\text{-}B)x_t = x_t\text{-}x_{t-1}$
  $$(1 - B)x_t = a_0 + e_t$$
- This is called an "integrated" model of order 1 or I(1). Since the errors are integrated to obtain x.
- Note that x_t is not stationary but (1-B)x_t is stationary.

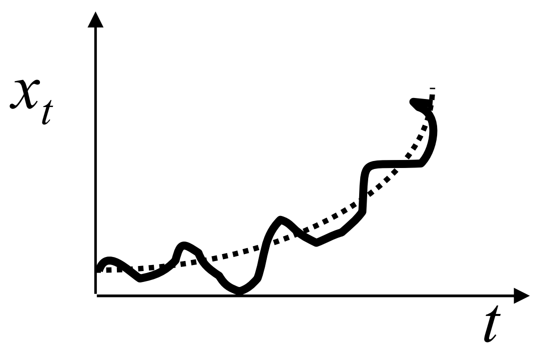




Integrated Models (Cont)

- If the time series is parabolic, the second difference can be modeled as white noise:
  $$(x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) = a_0 + e_t$$
- Or $(1 - B)^2 x_t = a_0 + e_t$
  This is an I(2) model. Also written as:
  $$D^2 x_t = b_0 + e_t$$

  Where Operator D = 1-B



ARMA and ARIMA Models

- It is possible to combine AR, MA, and I models
- ARMA(p, q) Model:

$$\begin{aligned} x_t - a_1 x_{t-1} - \ldots - a_p x_{t-p} &= b_0 + e_t + b_1 e_{t-1} + \ldots + b_q e_{t-q} \\ \phi_p(B) x_t &= b_0 + \psi_q(B) e_t \end{aligned}$$

- ARIMA(p,d,q) Model:

$$\phi_p(B)(1-B)^d x_t = b_0 + \psi_q(B) e_t$$

- Using algebraic manipulations, it is possible to transform AR models to MA models and vice versa.

Example 37.11

- Consider the MA(1) model: $x_t = b_0 + e_t + b_1 e_{t-1}$
- It can be written as: $(x_t - b_0) = (1 + b_1 B) e_t$

$$(1 + b_1 B)^{-1}(x_t - b_0) = e_t$$

$$\left(1 - b_1 B + b_1^2 B^2 - b_1^3 B^3 + \ldots\right)(x_t - b_0) = e_t$$

$$\left(x_t - b_1 x_{t-1} + b_1^2 x_{t-2} - b_1^3 x_{t-3} + \cdots\right) - \frac{b_0}{1 + b_1} = e_t$$

$$x_t = \frac{b_0}{1 + b_1} + b_1 x_{t-1} - b_1^2 x_{t-2} + b_1^3 x_{t-3} - \cdots + e_t$$

- If b_1<1, the coefficients decrease and soon become insignificant. This results in a finite order AR model.

Exercise 39.10

- Convert the following AR(1) model to an equivalent MA model:

$$x_t = a_0 + a_1 x_{t-1} + e_t$$

Non-Stationarity due to Seasonality

- The mean temperature in December is always lower than that in November and in May it always higher than that in March ⇒Temperature has a yearly season.
- One possible model could be I(12):

$$x_t - x_{t-12} = a_0 + e_t$$

- or

$$(1 - B^{12})x_t = a_0 + e_t$$

Seasonal ARIMA (SARIMA) Models

- SARIMA $(p, d, q) \times (P, R, Q)^s$ Model:

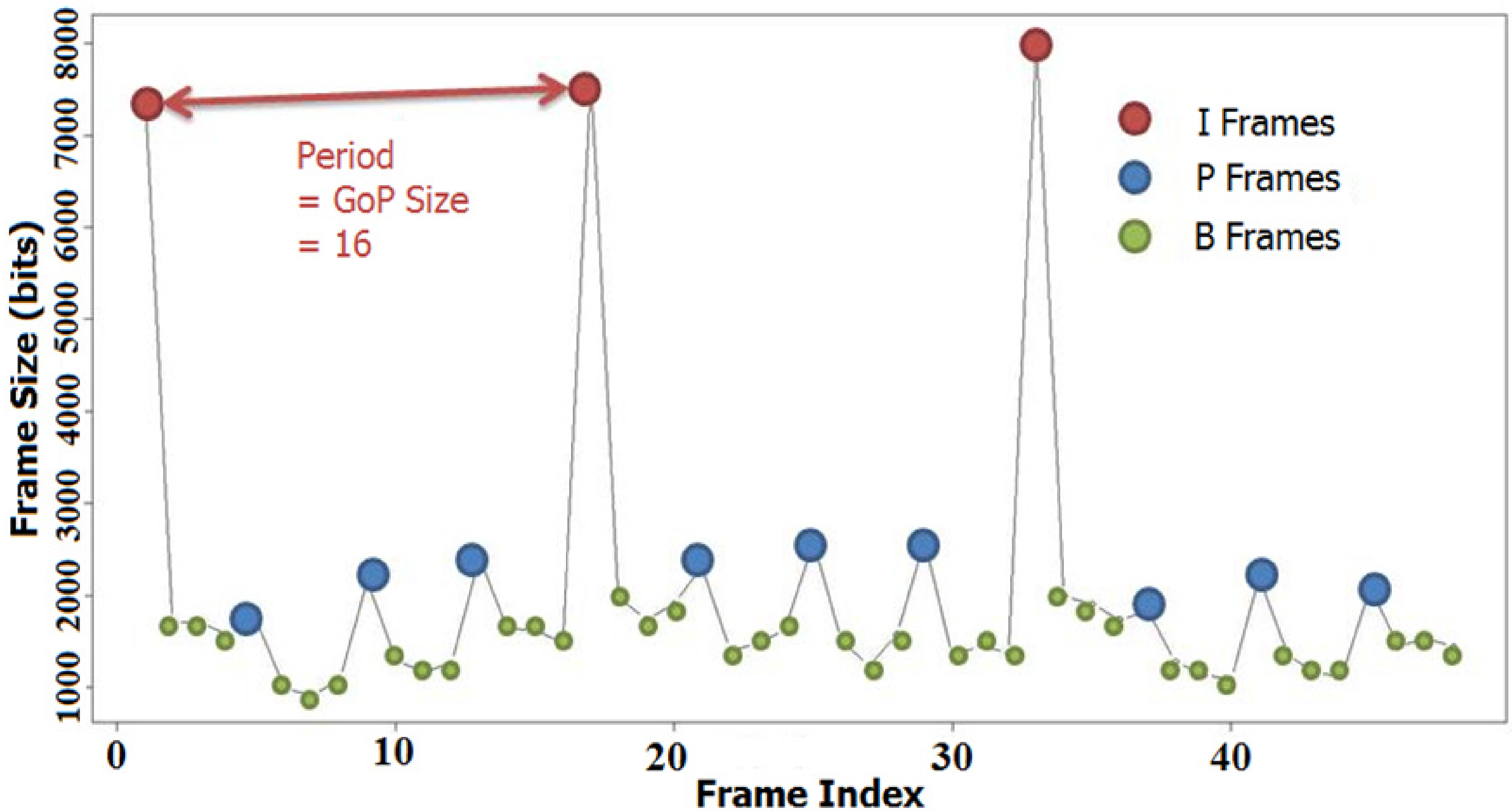
$$\phi_p(B)\Phi_P(B^s)(1-B^s)^R(1-B)^d x_t = b_0 + \psi_q(B)\Psi_Q(B^s)e_t$$

- Fractional ARIMA (FARIMA) Models
  ARIMA(p, d+δ, q) $-0.5 \leq \delta \leq 0.5$
  $\Rightarrow$Fractional Integration allowed.

Exercise 37.11

- Write the expression for SARIMA(1,0,1)(0,1,0)^{12} model in terms of x's and e's.

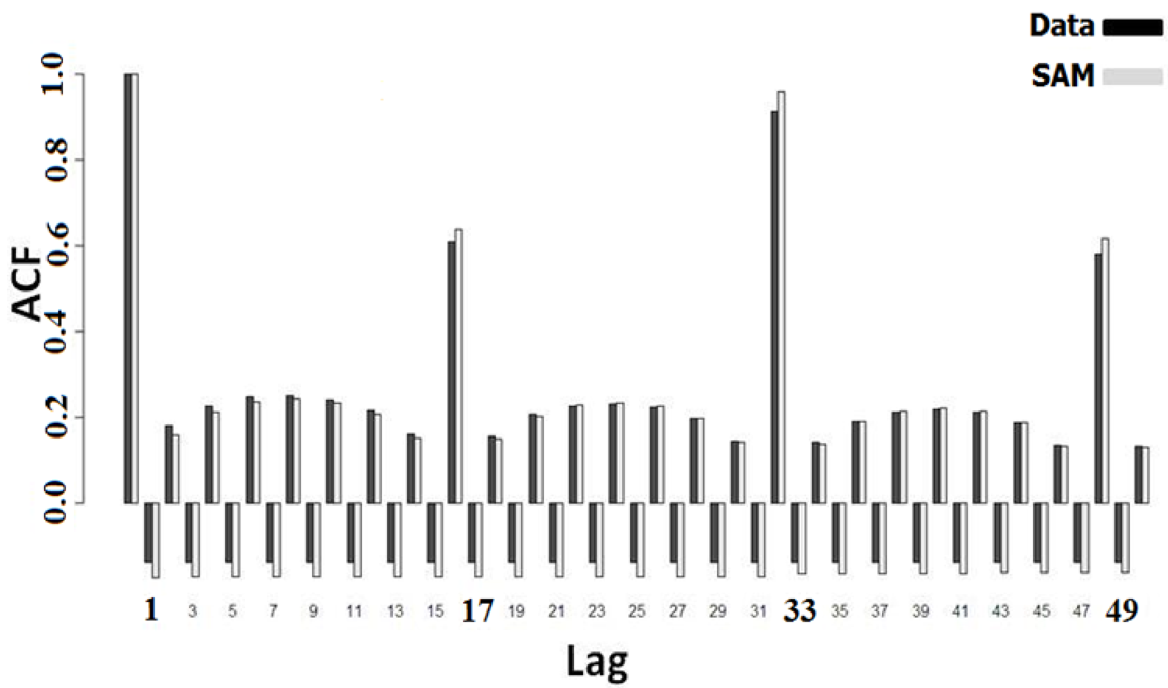
Case Study 37.1: Mobile Video



- Observation: Every 16th frame is a large (I) frame.

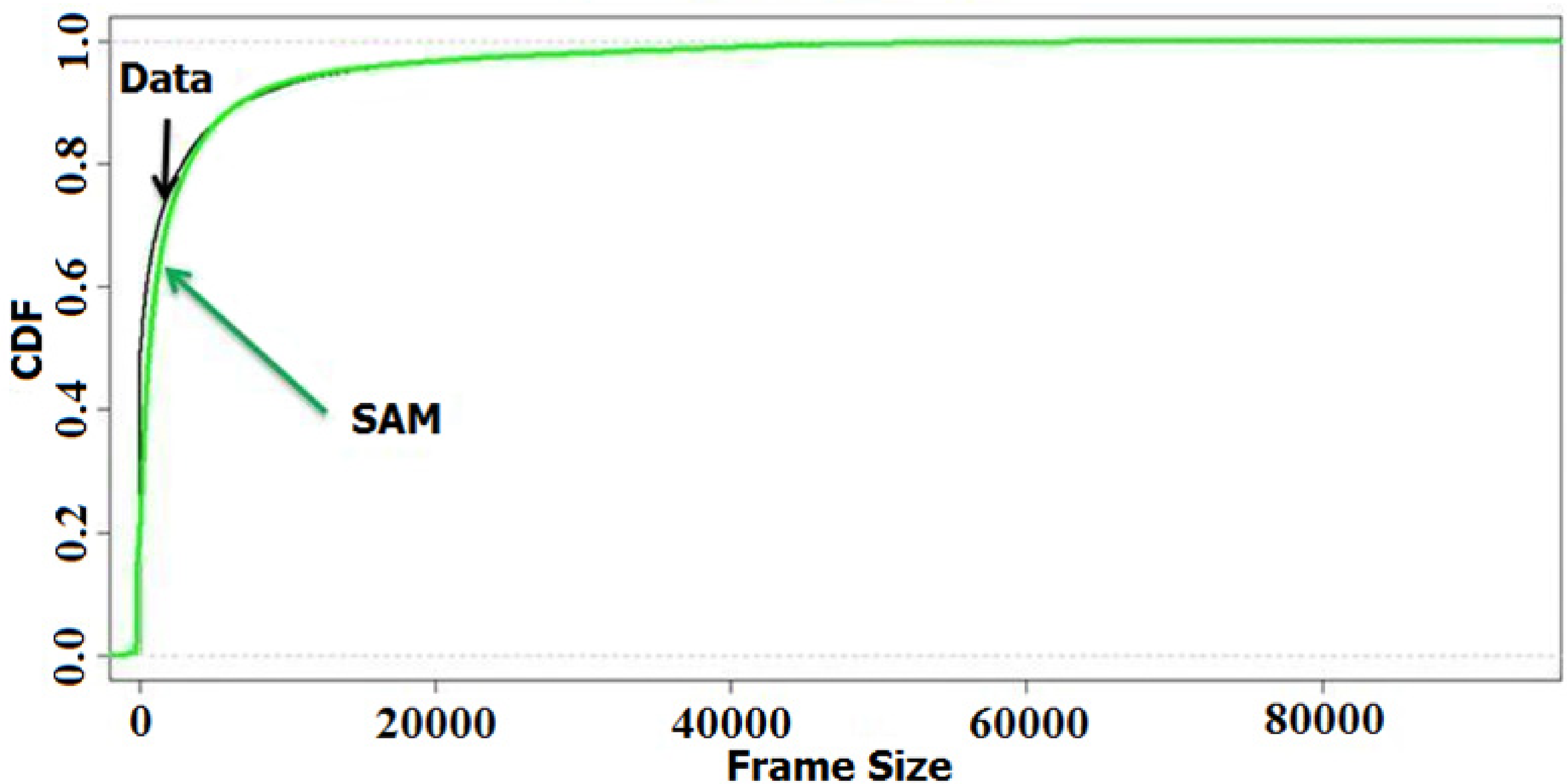
Traffic Modeling – All Frames

- A closer look at the ACF graph shows a strong continual correlation every 16 lag ➔ GOP size



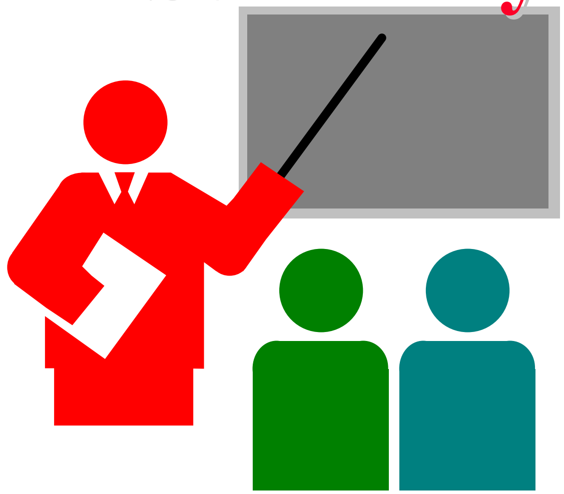
- Result: SARIMA (1, 0, 1)x(1,1,1)^s Model, s=group size =16

Validation



CDF
0.0
0.2
0.4
0.6
0.8
1.0
Data
SAM
0
20000
40000
60000
80000
Frame Size

Summary



- AR(1) Model:
  $$x_t = a_0 + a_1 x_{t-1} + e_t$$
- MA(1) Model:
  $$x_t - a_0 = e_t + b_1 e_{t-1}$$
- ARIMA(1,1,1) Model:
  $$x_t - x_{t-1} = a_0 + a_1(x_{t-1} - x_{t-2}) + e_t + b_1 e_{t-1}$$
- Seasonal ARIMA (1,0,1)x(0,1,0)^{12} model:
  $$x_t - x_{t-12} = a_0 + a_1(x_{t-1} - x_{t-13}) + e_t + b_1 e_{t-1}$$