2^{k-p} Fractional Factorial Designs

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- 2^{k-p} Fractional Factorial Designs
- Sign Table for a 2^{k-p} Design
- Confounding
- Other Fractional Factorial Designs
- Algebra of Confounding
- Design Resolution

2^{k-p} Fractional Factorial Designs

- Large number of factors
  $\Rightarrow$ large number of experiments
  $\Rightarrow$ full factorial design too expensive
  $\Rightarrow$ Use a fractional factorial design
- 2^{k-p} design allows analyzing k factors with only 2^{k-p} experiments.
  2^{k-1} design requires only half as many experiments
  2^{k-2} design requires only one quarter of the experiments

Example: 2^{7-4} Design

Expt No.	A	B	C	D	E	F	G
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

- Study 7 factors with only 8 experiments!

Fractional Design Features

- Full factorial design is easy to analyze due to orthogonality of sign vectors.

  Fractional factorial designs also use orthogonal vectors. That is:
  - The sum of each column is zero.

    $$\sum_i x_{ij} = 0 \quad \forall j$$

  *j*th variable, *i*th experiment.
  - The sum of the products of any two columns is zero.

    $$\sum_i x_{ij} x_{il} = 0 \quad \forall j \neq l$$
  - The sum of the squares of each column is 2^{7-4}, that is, 8.

    $$\sum_i x_{ij}^2 = 8 \quad \forall j$$

Analysis of Frac. Factorial Designs

- **Model**:

$$y = q_0 + q_A x_A + q_B x_B + q_C x_C + q_D x_D + q_E x_E + q_F x_F + q_G x_G$$

- Effects can be computed using inner products.

$$q_A = \sum_i y_i x_{Ai} = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}$$

$$q_B = \sum_i y_i x_{Bi} = \frac{-y_1 - y_2 + y_3 + y_4 - y_5 - y_6 + y_7 + y_8}{8}$$

Example 19.1

I	A	B	C	D	E	F	G	y
1	-1	-1	-1	1	1	1	-1	20
1	1	-1	-1	-1	-1	1	1	35
1	-1	1	-1	-1	1	-1	1	7
1	1	1	-1	1	-1	-1	-1	42
1	-1	-1	1	1	-1	-1	1	36
1	1	-1	1	-1	1	-1	-1	50
1	-1	1	1	-1	-1	1	-1	45
1	1	1	1	1	1	1	1	82
317	101	35	109	43	1	47	3	Total
39.62	12.62	4.37	13.62	5.37	0.125	5.87	0.37	Total/8

- Factors A through G explain 37.26%, 4.47%, 43.40%, 6.75%, 0%, 8.06%, and 0.03% of variation, respectively.

  ⇒ Use only factors C and A for further experimentation.

Sign Table for a 2^{k-p} Design

Steps:

1. Prepare a sign table for a full factorial design with k-p factors.
2. Mark the first column I.
3. Mark the next k-p columns with the k-p factors.
4. Of the $(2^{k-p}-k+p-1)$ columns on the right, choose p columns and mark them with the p factors which were not chosen in step 1.

Example: 2^{7-4} Design

Expt No.	A	B	C	AB	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

Example: 2^{4-1} Design

Expt No.	A	B	C	AB	AC	BC	D
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

Confounding

- **Confounding**: Only the combined influence of two or more effects can be computed.

$$q_A = \sum_i y_i x_{Ai} = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}$$

$$q_D = \sum_i y_i x_{Di} = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

Confounding (Cont)

$$q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci} = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

$$q_D = q_{ABC}$$

$$q_D + q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci} = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

- $\Rightarrow$ Effects of D and ABC are confounded. Not a problem if q_{ABC} is negligible.

Confounding (Cont)

- Confounding representation: *D=ABC*
  Other Confoundings:

$$q_A = q_{BCD} = \sum_i y_i x_{Ai}$$
$$= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}$$

$\Rightarrow A = BCD$

A=BCD, B=ACD, C=ABD, AB=CD, AC=BD, BC=AD, ABC=D, and I=ABCD

- *I=ABCD* ⇒ confounding of ABCD with the mean.

Other Fractional Factorial Designs

- A fractional factorial design is not unique. 2^p different designs.

Another 2^{4-1} Experimental Design

Expt No.	A	B	C	D	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

- Confoundings: I=ABD, A=BD, B=AD, C=ABCD, D=AB, AC=BCD, BC=ACD, ABC=CD

Not as good as the previous design.

Algebra of Confounding

- Given just one confounding, it is possible to list all other confoundings.
- Rules:
  - *I* is treated as unity.
  - Any term with a power of 2 is erased.

$$I = ABCD$$

Multiplying both sides by A:

$$A = A^2BCD = BCD$$

Multiplying both sides by B, C, D, and AB:

Algebra of Confounding (Cont)

$$B = AB^2CD = ACD$$
$$C = ABC^2D = ABD$$
$$D = ABCD^2 = ABC$$
$$AB = A^2B^2CD = CD$$

and so on.

- Generator polynomial: *I=ABCD*

For the second design: *I=ABC*.

- In a 2^{k-p} design, 2^p effects are confounded together.

Example 19.7

- In the 2^{7-4} design:
  D = AB, E = AC, F = BC, G = ABC

$\Rightarrow$ I = ABD, I = ACE, I = BCF, I = ABCG

$\Rightarrow$ I = ABD = ACE = BDF = ABCG

- Using products of all subsets:

				D	E	F	G
Expt No.	A	B	C	AB	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

I = ABD = ACE = BCF = ABCG
= ABD*ACE = ABD*BCF = ABD*ABCG
= ACE*BCF = ACE*ABCG
= BCF*ABCG
} Two at a time

=ABD*ACE*BCF=ABC*ACE*ABCG=ABD*BCF*AGCG=ACE*BCF*ABCG ← Three at a time

= ABD*ACE*BCF*ABCG } Four at a time

$\Rightarrow$ I = ABD = ACE = BCF = ABCG
= BCDE = ACDF = CDG
= ABEF = BEG
= AFG
= DEF = ACEG = DFG = CEFG
= ABCDEFG

Example 19.7 (Cont)

- I = ABD = ACE = BCF = ABCG = BCDE = ACDF = CDG
  = ABEF = BEG = AFG = DEF = ACEG = DFG = CEFG
  = ABCDEFG
- Total 2^p terms in the generator polynomial
- Other confoundings: Multiply both sides above by A
  A = BD = CE = ABCF = BCG = CDE = CDF = ACDG
  = BEF = ABEG = FG = ADEF = CEG = ADFG = ACEFG
  = BCDEFG

Design Resolution

- Order of an effect = Number of terms
  Order of *ABCD* = 4, order of I = 0.
- Order of a confounding = Sum of order of two terms
  E.g., AB=CDE is of order 5.
- Resolution of a Design
  = Minimum of orders of confoundings
- Notation: R_{III} = Resolution-III = 2^{k-p}_{III}
- Example 1: *I=ABCD* $\Rightarrow$ R_{IV} = Resolution-IV = 2^{4-1}_{IV}
  A=BCD, B=ACD, C=ABD, AB=CD, AC=BD,
  BC=AD, ABC=D, and I=ABCD

Design Resolution (Cont)

- Example 2:
  $I = ABD \Rightarrow R_{III}$ design.
- Example 3:

$$\begin{aligned} I &= ABD = ACE = BCF = ABCG = BCDE \\ &= ACDF = CDG = ABEF = BEG \\ &= AFG = DEF = ADEG = BDFG \\ &= CEFG = ABCDEFG \end{aligned}$$

- This is a resolution-III design.
- A design of higher resolution is considered a better design.

Number of Fractional Factorial Designs

- Table contains 2^{k-p} columns.
- Of these (k-p)+1 are main effects
- Choices for the next main effect:

$$m = 2^{k-p} - (k-p) - 1$$

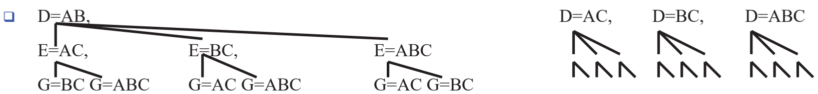
- Choices for the next main effect: *m-1*

  ....

Expt No.	A	B	C	AB	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

- Total number of designs = $m(m-1)(m-2)\ldots(m-p+1) = \frac{\lfloor m}{\lfloor m-p}$
- **Example**: 2^{7-4} Design. $m=2^3-3-1=4$

Number of possible designs $= \frac{m!}{(m-p)!} = \frac{4!}{1!} = 24$



Case Study 19.1: Latex vs. troff

Factors and Levels

	Factor	-Level	+Level
A	Program	Latex	troff-me
B	Bytes	2100	25000
C	Equations	0	10
D	Floats	0	10
E	Tables	0	10
F	Footnotes	0	10

Case Study 19.1 (Cont)

- Design: 2^{6-1} with I=BCDEF

Factor		Effect	% Variation
B	Bytes	12.0	39.4%
A	Program	9.4	24.4%
C	Equations	7.5	15.6%
AC	Program × Equations	7.2	14.4%
E	Tables	3.5	3.4%
F	Footnotes	1.6	0.70%

Case Study 19.1: Conclusions

- Over 90% of the variation is due to: Bytes, Program, and Equations and a second order interaction.
- Text file size were significantly different making it's effect more than that of the programs.
- High percentage of variation explained by the ``program × Equation'' interaction
  ⇒ Choice of the text formatting program depends upon the number of equations in the text. troff not as good for equations.

CPU Time

Program	# of Equations	
	-1(0)	1(10)
-1(Latex)	-9.7	-9.1
1(Troff)	-5.3	24.1

Case Study 19.1: Conclusions (Cont)

- Low ``Program × Bytes" interaction ⇒ Changing the file size affects both programs in a similar manner.
- In next phase, reduce range of file sizes. Alternately, increase the number of levels of file sizes.

Summary

- Fractional factorial designs allow a large number of variables to be analyzed with a small number of experiments
- Many effects and interactions are confounded
- The resolution of a design is the sum of the order of confounded effects
- A design with higher resolution is considered better

Homework 19

- **Updated** Exercise 19.1 Analyze the 2^{4-1} design:

		C_1		C_2	
		D_1	D_2	D_1	D_2
A_1	B_1		30	15	
	B_2		20	10	
A_2	B_1	100			30
	B_2	110			50

- Quantify all main effects.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.
- List all confoundings.
- Can you propose a better design with the same number of experiments.
- What is the resolution of the design?

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