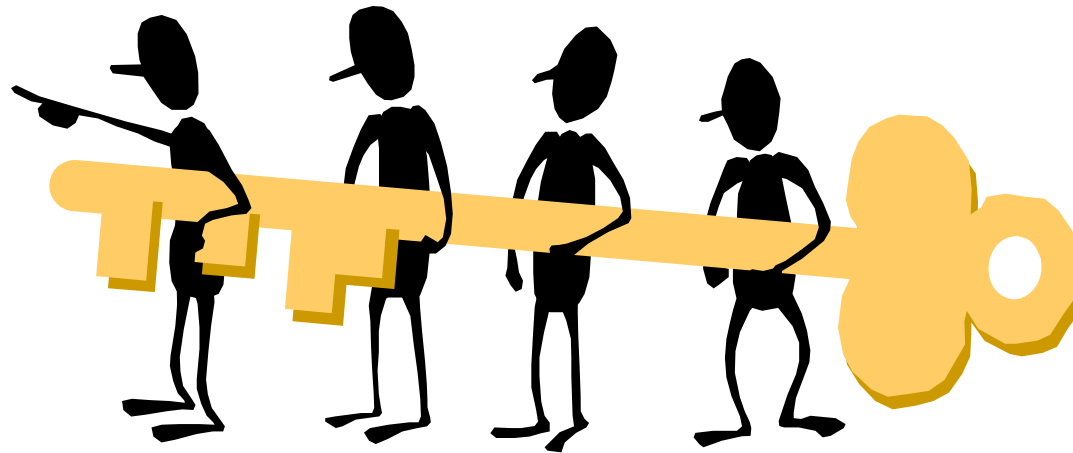


Public Key Algorithms



Raj Jain

Washington University in Saint Louis

Saint Louis, MO 63130

Jain@cse.wustl.edu

Audio/Video recordings of this lecture are available at:

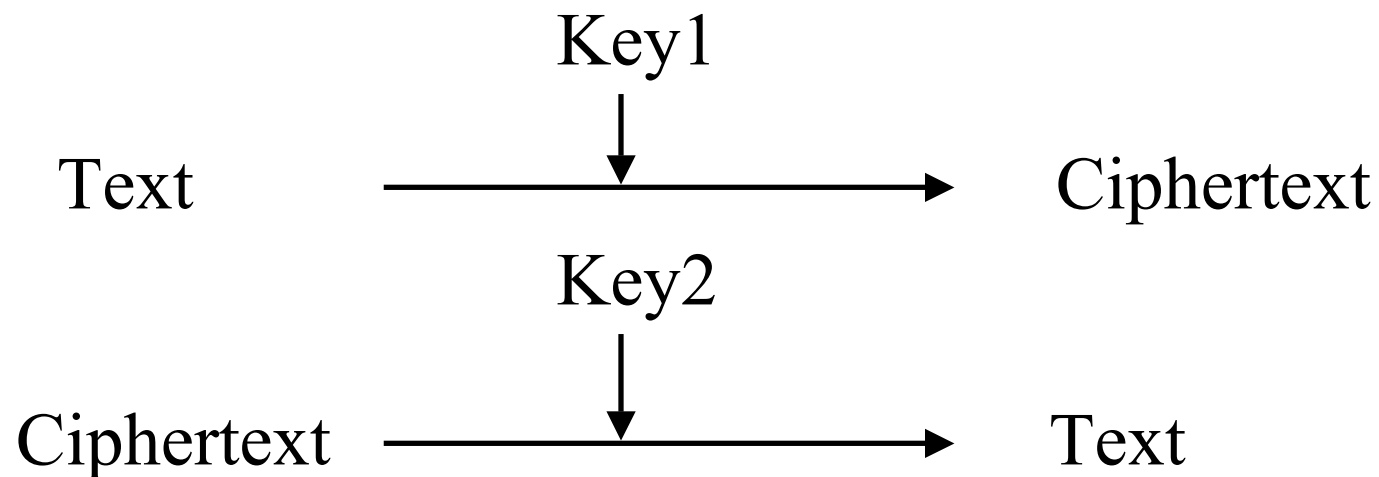
<http://www.cse.wustl.edu/~jain/cse571-07/>



1. Number Theory
2. RSA Public Key Encryption
3. Public-Key Cryptography Standards (PKCS)
4. Diffie-Hellman Key Agreement
5. Digital Signature Standard
6. Elliptic Curve Cryptography (ECC)
7. Zero-Knowledge Proof Systems

Public Key Encryption

- ❑ Invented in 1975 by Diffie and Hellman
- ❑ $\text{Encrypted_Message} = \text{Encrypt}(\text{Key1}, \text{Message})$
- ❑ $\text{Message} = \text{Decrypt}(\text{Key2}, \text{Encrypted_Message})$



Public Key Encryption Example

- ❑ Rivest, Shamir, and Adleman
- ❑ RSA: Encrypted_Message = $m^3 \bmod 187$
- ❑ Message = Encrypted_Message¹⁰⁷ mod 187
- ❑ Key1 = <3,187>, Key2 = <107,187>
- ❑ Message = 5
- ❑ Encrypted Message = $5^3 = 125$
- ❑ Message = $125^{107} \bmod 187 = 5$
= $125^{(64+32+8+2+1)} \bmod 187$
= $\{(125^{64} \bmod 187)(125^{32} \bmod 187) \dots$
 $(125^2 \bmod 187)(125 \bmod 187)\} \bmod 187$

Modular Arithmetic

- $xy \bmod m = (x \bmod m)(y \bmod m) \bmod m$
- $x^4 \bmod m = (x^2 \bmod m)(x^2 \bmod m) \bmod m$
- $x^{ij} \bmod m = (x^i \bmod m)^j \bmod m$
- $125 \bmod 187 = 125$
- $125^2 \bmod 187 = 15625 \bmod 187 = 104$
- $125^4 \bmod 187 = (125^2 \bmod 187)^2 \bmod 187$
 $= 104^2 \bmod 187 = 10816 \bmod 187 = 157$
- $128^8 \bmod 187 = 157^2 \bmod 187 = 152$
- $128^{16} \bmod 187 = 152^2 \bmod 187 = 103$
- $128^{32} \bmod 187 = 103^2 \bmod 187 = 137$
- $128^{64} \bmod 187 = 137^2 \bmod 187 = 69$
- $128^{64+32+8+2+1} \bmod 187 = 69 \times 137 \times 152 \times 104 \times 125 \bmod 187$
 $= 18679128000 \bmod 187 = 5$

Definitions

- ❑ Co-Prime = Relatively Prime:
x and y are relatively prime if $\gcd(x,y)=1$
 - Example: 6, 11
- ❑ Inverse: x is inverse of y if $xy=1 \pmod n$
 - If $ux + vn = 1$, $x^{-1} = u \pmod n$
 - Example: what is inverse of 6 mod 11
 - ❑ $2 \times 6 - 1 \times 11 = 1 \Rightarrow$ Inverse of 6 mod 11 is 2.
- ❑ Smooth Number = Product of small primes

Euclid's Algorithm

□ Goal: To find greatest common divisor

Example: $\text{gcd}(10,25)=5$ using long division

10) 25 (2

20

--

5)10 (2

10

--

00

Euclid's Algorithm: Tabular Method

		10	25
q_i	r_i	u_i	v_i
0	25	0	1
0	10	1	0
2	5	-2	1
2	0	5	-2

- $r_i = u_i x + v_i y$
- $u_i = u_{i-2} - q_i u_{i-1}$
- $v_i = v_{i-2} - q_i v_{i-1}$
- Finally, If $r_i = 0$, $\gcd(x,y) = r_{i-1}$
- If $r_i = 1$, $u_i x + v_i y = 1 \Rightarrow x^{-1} \bmod y = u_i$

Chinese Remainder Theorem

- The solution to the following equations:

$$x = a_1 \pmod{n_1}$$

$$x = a_2 \pmod{n_2}$$

$$x = a_k \pmod{n_k}$$

where n_1, n_2, \dots, n_k are relatively prime is found as follows:

$$N = n_1 n_2 \dots n_k$$

$$N_i = N/n_i$$

Find s_i such that $r_i n_i + s_i N_i = 1$

Let $e_i = s_i N_i$, then

$$x = \sum_i^k a_i e_i \pmod{N}$$

Chinese Remainder Theorem (Cont)

□ Example: Solve the equations:

□ $x = 3 \pmod{6}$

□ $x = 6 \pmod{7}$

□ $x = 10 \pmod{11}$

□ $N = 6 \times 7 \times 11 = 462$

□ $\gcd(6,77): 13 \times 6 - 1 \times 77 = 1 \Rightarrow e_1 = -77$

□ $\gcd(7,66): 19 \times 7 - 2 \times 66 = 1 \Rightarrow e_2 = -132$

□ $\gcd(11,42): -19 \times 11 + 5 \times 42 = 1 \Rightarrow e_3 = 210$

□ $x = 3 \times (-77) + 6 \times (-132) + 10 \times 210 = -231 - 792 + 2100 = 1077 \pmod{462} = 153$

	7	66		
q_i	r_i	u_i	v_i	
0	66	0	1	
0	7	1	0	
9	3	-9	1	
2	1	19	-2	

Euler's Totient Function

- $Z_n =$ Set of all numbers mod $n = \{0, 1, 2, \dots, n-1\}$
- $Z_n^* =$ Set of all numbers relatively primes to n
- $\Phi(n) =$ Number of elements in Z_n^*
- If n is prime, $\Phi(n)=n-1$
- Example:

$$Z_{10} = \{0, 1, 2, 3, \dots, 9\}$$

$$Z_{10}^* = \{1, 3, 7, 9\}$$

$$\Phi(10)=4$$

Euler's Theorem

- For all $a \in Z_n^*$ $a^{\Phi(n)} = 1 \pmod n$
- For all $a \in Z_n^*$ $a^{(k\Phi(n)+1)} = a \pmod n$
- Examples:
 - $Z_{10}^* = \{1, 3, 7, 9\}$
 - $\Phi(10)=4$
 - $1^4 \pmod{10} = 1$
 - $3^4 \pmod{10} = 1$
 - $7^4 \pmod{10} = 1$
 - $9^4 \pmod{10} = 1$

Fermat's Theorem

- ❑ If p is prime and $0 < a < p$, $a^{p-1} \bmod p = 1$
- ❑ Example:
 - $2^6 \bmod 7 = 64 \bmod 7 = 1$
 - $3^4 \bmod 5 = 81 \bmod 5 = 1$
 - This is sufficient condition. Not necessary.
 - $a^{p-1} \bmod p = 1$ for all $a \neq 0 \Rightarrow p$ is prime
 - **Carmichael Numbers** or pseudo-primes
 - Example: $561 = 3 \times 11 \times 17$

Miller and Rabin Method Prime Test

- ❑ Express $n-1$ as $2^b c$ where c is odd.
- ❑ Pick a random a .
- ❑ $a^{n-1} = a^{2^b c} = (a^c)^{2^b}$
- ❑ Compute $a^c \bmod n$
- ❑ Square it b times: $a^{2c}, a^{4c}, a^{8c}, \dots, a^{\{2^b\}c}$
- ❑ If the final result is not one $\Rightarrow n$ is not a prime
- ❑ If any of the intermediate results is 1, check if the previous number is ± 1 .
- ❑ If ± 1 then n is potentially prime.
- ❑ Pick another a and try again.

RSA Public Key Encryption

- ❑ Ron Rivest, Adi Shamir, and Len Adleman at MIT 1978
- ❑ Both plain text M and cipher text C are integers between 0 and $n-1$.
- ❑ Key 1 = $\{e, n\}$,
Key 2 = $\{d, n\}$
- ❑ $C = M^e \bmod n$
 $M = C^d \bmod n$
- ❑ How to construct keys:
 - Select two large primes: $p, q, p \neq q$
 - $n = p \times q$
 - Calculate Euler's Totient $\Phi(n) = (p-1)(q-1)$
 - Select e relatively prime to $\Phi \Rightarrow \gcd(\Phi, e) = 1; 0 < e < \Phi$
 - Calculate $d = \text{inverse of } e \bmod \Phi \Rightarrow de \bmod \Phi = 1$
 - Euler's Theorem: $x^{ed} = x^{k\Phi(n)+1} = x \bmod n$

RSA Key Construction: Example

- ❑ Select two large primes: $p, q, p \neq q$
 $p = 17, q = 11$
- ❑ $n = p \times q = 17 \times 11 = 187$
- ❑ Calculate $\Phi = (p-1)(q-1) = 16 \times 10 = 160$
- ❑ Select e , such that $\text{lcd}(\Phi, e) = 1; 0 < e < \Phi$
say, $e = 7$
- ❑ Calculate d such that $de \text{ mod } \Phi = 1$
 - $160k+1 = 161, 321, 481, 641$
 - Check which of these is divisible by 7
 - 161 is divisible by 7 giving $d = 161/7 = 23$
- ❑ Key 1 = $\{7, 187\}$, Key 2 = $\{23, 187\}$

RSA Issues

- ❑ RSA is computationally intense.
- ❑ Commonly used key lengths are 512 bits
- ❑ The plain text should be smaller than the key length
- ❑ The encrypted text is same size as the key length
- ❑ Generally used to encrypt secret keys.
- ❑ Basis: Factoring a big number is hard

Finding d and e

- ❑ $de = 1 \pmod{\Phi(n)}$
- ❑ Select e first, e.g., $e=2^1+1$ or $2^{16}+1$
⇒ Exponentiation is easy.
- ❑ Find inverse of e using Euler's algorithm
- ❑ The public key can be small.
- ❑ The private key should be large. ⇒ Don't select $d=3$.
- ❑ Both d and n are 512 bit (150 digits) numbers.

Optimizing Private Key Operations

1. $c^d \bmod n = c^d \bmod pq$
 - Compute $c^d \bmod p$ and $c^d \bmod q$
 - Use Chinese remainder theorem to compute $c^d \bmod pq$
2. Chinese remainder theorem requires $p^{-1} \bmod q$ and $q^{-1} \bmod p$. Compute them once and store.
3. Since d is much bigger than p , $c^d \bmod p = c^r \bmod p$ where $r = d \bmod (p-1)$
 - $d = k(p-1) + r$
 - Mod p : $a^d = a^{k(p-1)+r} = a^{k(\Phi(p))} a^r = a^r$ [Euler's Theorem]

Attacks on RSA

□ Smooth Number Attack:

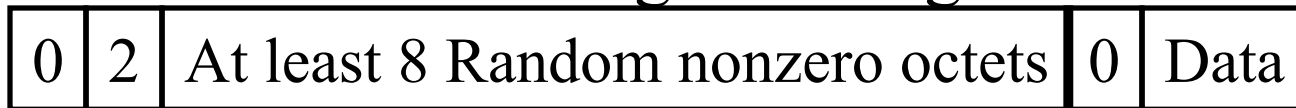
- If you sign m_1 and m_2
- $S_1 = m_1^d \bmod n$
- $S_2 = m_2^d \bmod n$
- Attacker can sign $m_1 m_2$, m_1/m_2 , m_1^2 , $m_1^j m_2^k$
- Easy to do if m_i 's are small (smooth) numbers.

□ Cube Root Problem of RSA

- If public exponent $e=3$:
- $h^{de} \bmod n = h$
- $h^d \bmod n = h^{1/3}$
- Simply compute $h^{1/3} \bmod n$

Public-Key Cryptography Standards

- ❑ RSA Inc developed standards on how to use public key cryptography
- ❑ Specify encoding of keys, signatures, etc.
- ❑ PKCS #1: Formatting a message for RSA encryption



- ❑ First octet = 0 \Rightarrow $m < n$
- ❑ Second Octet = Format Type. 2 \Rightarrow Encryption
- ❑ Random non-zero padding \Rightarrow cipher is different
- ❑ Zero ends the padding
- ❑ PKCS Signing

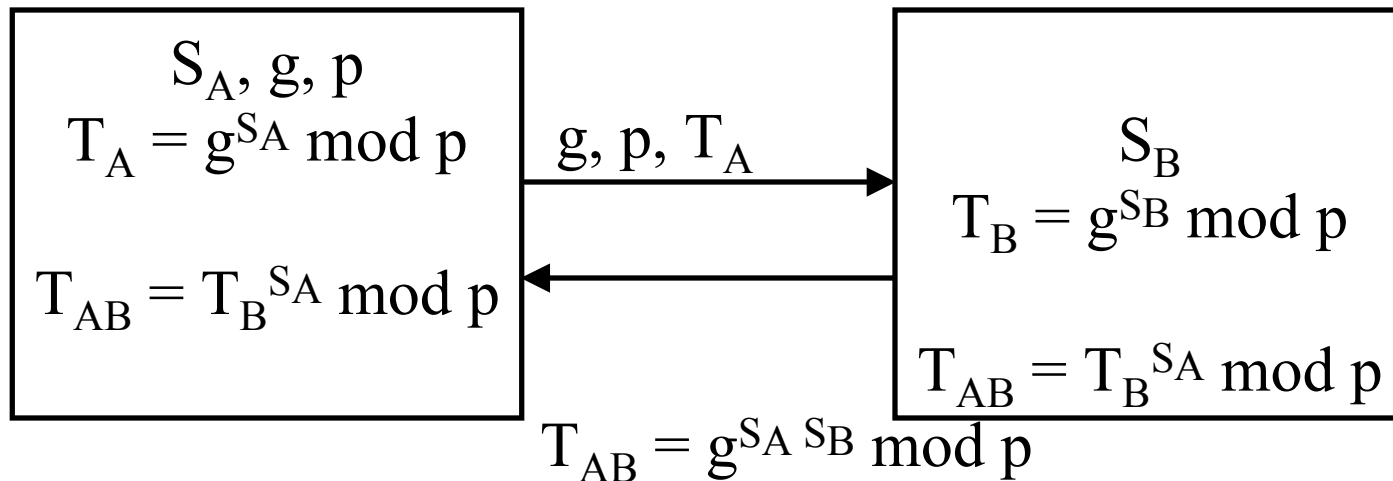


Million Message Attack on RSA

- ❑ In SSL if padding is incorrect, some Servers send "Incorrect Padding" error
- ❑ Attacker sends random variations until server accepts (decrypted message begins with 02)
- ❑ Would need 2^{16} tries to get the correct 16 bits
- ❑ PKCS#1 Rev 2 fixes this problem.
- ❑ Not sending error message is easier fix.

Diffie-Hellman Key Agreement

- Allows two party to agree on a secret key using a public channel



- Eavesdropper can see T_A, g, p but cannot compute S_A
- Computing S_A requires discrete logarithm - a difficult problem

Diffie-Hellman (Cont)

- ❑ Example: $g=5$, $p=19$
 - A selects 6 and sends $5^6 \bmod 19 = 7$
 - B selects 7 and sends $5^7 \bmod 19 = 16$
 - A computes $K = 16^6 \bmod 19 = 7$
 - B computes $K = 7^7 \bmod 19 = 7$
- ❑ Preferably $(p-1)/2$ should also be a prime.
- ❑ Such primes are called safe prime.

ElGamal Signatures

- ❑ Similar to Diffie-Hellman
- ❑ Public key: (g, p, T) , $T = g^S \pmod p$; Private key: S

Signature:

- ❑ Choose a random S_m , $0 < S_m < p-1$ and $\gcd(S_m, p-1) = 1$
- ❑ Compute $T_m = g^{S_m} \pmod p$
- ❑ Compute $X = (H(m|T_m)S_m + T_m) \pmod{(p-1)}$
- ❑ If $X = 0$ start over again
- ❑ The pair T_m, X is the signature.

Verification: Compute $H(m|T_m)$ and g^X and verify:

- ❑ $g^X = T_m T^{H(m|T_m)}$ (since $T = g^S$)
- ❑ Note: Each message needs a different per message key S_m .
- ❑ If the same key is used on many messages, S can be obtained.

Digital Signature Standard

- ❑ FIPS 186 in 1991, 186-1 in 1993, 186-2 in 2000.
- ❑ A variation of ElGamal signature
- ❑ Choose a hash. Default = SHA-1
- ❑ Select a key size L: multiple of 64 between 512 to 1024.
- ❑ 186-2 requires 1024.
- ❑ 186-3 recommends 2048 or 3072 for lifetimes beyond 2010.

1. Algorithm Parameters:

- ❑ Choose a prime q with the same number of bits as hash
- ❑ Select a L -bit prime p such that $p-1$ is a multiple of q
- ❑ Select a generator g such that $g^q = 1 \pmod{p}$
- ❑ This can be done by $g = h^{(p-1)/q} \pmod{p}$ for some arbitrary h
 $1 < h < p-1$.
- ❑ Algorithm parameters (p, q, g) may be shared among users.

DSS (Cont)

2. User Keys: public and private key for a user

- ❑ Choose S randomly $0 < S < p$
- ❑ $T = g^S \text{ mod } p$
- ❑ Public key is (p, q, g, T) . Private key is S .

3. Signing: Generate per message key S_m , $0 < S_m < q$

- ❑ $T_m = (g^{S_m} \text{ mod } p) \text{ mod } q$
- ❑ Compute $S_m^{-1} \text{ mod } q$
- ❑ Calculate message digest d_m
- ❑ Signature $X = S_m^{-1} (d_m + ST_m) \text{ mod } q$
- ❑ Transmit message m , per message public number T_m , and signature X

DSS (Cont)

4. Verification:

- ❑ Calculate inverse of signature $X^{-1} \bmod q$
- ❑ Calculate message digest d_m
- ❑ Calculate $x = d_m X^{-1} \bmod q$
- ❑ $y = T_m x^{-1} \bmod q$
- ❑ $z = (g^x T^y \bmod p) \bmod q$
- ❑ If $z = T_m$ then signature is verified.

DSS Insecurity

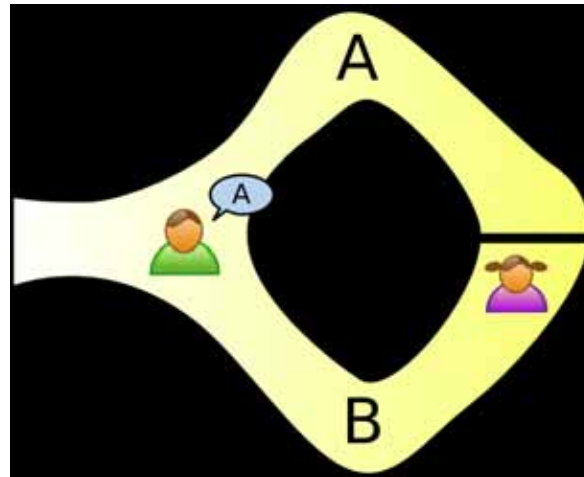
- ❑ $\langle p, q, g \rangle$ is shared.
- ❑ Anyone who breaks $\langle p, q, g \rangle$ can break all the users sharing it.
- ❑ Slower than RSA with $e=3$
- ❑ Both RSA and Diffie-Hellman require sub-exponential super-polynomial effort

Elliptic Curve Cryptography (ECC)

- ❑ Based on algebraic structure of elliptic curves over finite fields
- ❑ Elliptic curve is a plane curve
- ❑ $y^2 + axy + by = x^3 + cx^2 + dx + e$
- ❑ The set of points on the curve along with the point at infinity form a set over which operations similar to modular arithmetic can be defined.
- ❑ Multiplying two points results in a third point. The point at infinity is the identity element.
- ❑ ECC is still exponential difficulty and so key lengths can be shorter.

Zero-Knowledge Proof Systems

- ❑ The verifier can verify that you possess the secret but gets no knowledge of the secret.

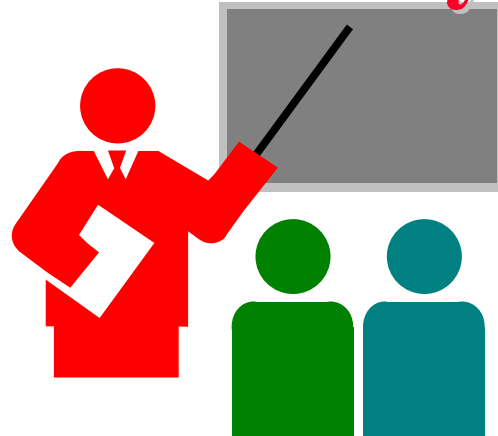


[Source: Wikipedia]

Any NP-complete problem can be used.

- ❑ In graph theory, if you know Hamiltonian cycle of a large graph you can easily generate isomorphic graphs.

Summary



- ❑ Public key cryptography uses two keys: public key and private key
- ❑ Modular Arithmetic, Euler's algorithm, Euler's theorem, Fermat's theorem, Chinese remainder theorem
- ❑ RSA is based on difficulty of factorization
- ❑ Diffie-Hellman is based on difficulty of discrete logarithms.
- ❑ Digital signature standard is similar to Diffie-Hellman

References

- ❑ Chapter 6 and 7 of the text book
- ❑ Wikipedia entries:
 - http://en.wikipedia.org/wiki/Extended_Euclidean_algorithm
 - http://en.wikipedia.org/wiki/Chinese_remainder_theorem
 - http://en.wikipedia.org/wiki/Carmichael_number
 - <http://en.wikipedia.org/wiki/PKCS>
 - <http://en.wikipedia.org/wiki/Diffie-Hellman>
 - http://en.wikipedia.org/wiki/ElGamal_signature_scheme
 - http://en.wikipedia.org/wiki/Digital_Signature_Standard
 - http://en.wikipedia.org/wiki/Elliptic_curve_cryptography
 - http://en.wikipedia.org/wiki/Zero_Knowledge

Homework 8

- ❑ Read chapter 6 and 7
- ❑ 8a. In an RSA system, the public key of a given user is $e=31$, $n=3599$. What is the private key of this user.
- ❑ 8b. If $x = 3 \pmod{7} = 5 \pmod{13} = 8 \pmod{11}$. What is x ? Show all steps.