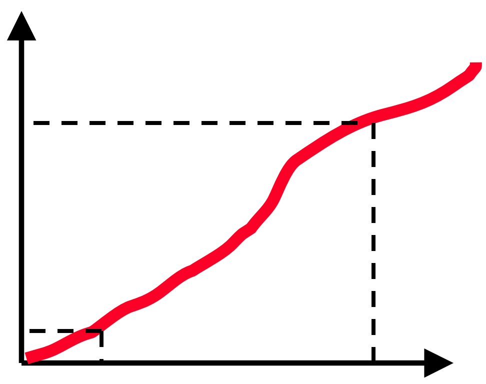
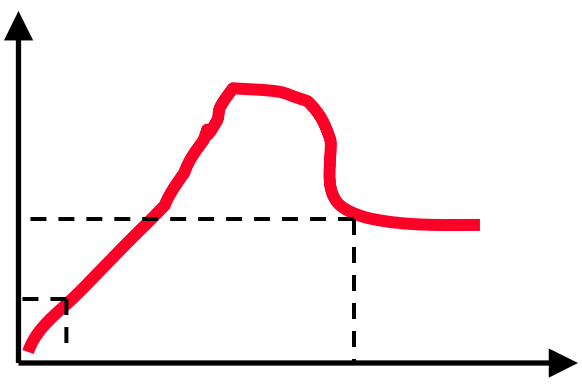
# $2^k$ Factorial Designs


- $2^2$ Factorial Designs
- Model
- Computation of Effects
- Sign Table Method
- Allocation of Variation
- General $2^k$ Factorial Designs

# $2^k$ Factorial Designs

- k factors, each at two levels.
- Easy to analyze.
- Helps in sorting out impact of factors.
- Good at the beginning of a study.
- Valid only if the effect is unidirectional.
  E.g., memory size, the number of disk drives





# $2^2$ Factorial Designs

- Two factors, each at two levels.

Performance in MIPS

| Cache Size | Memory Size | |
|---|---|---|
| | 4M Bytes | 16M Bytes |
| 1K | 15 | 45 |
| 2K | 25 | 75 |

$$x_A = \begin{cases} -1 & \text{if 4M bytes memory} \\ 1 & \text{if 16M bytes memory} \end{cases}$$

$$x_B = \begin{cases} -1 & \text{if 1K bytes cache} \\ 1 & \text{if 2K bytes cache} \end{cases}$$

# Model

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

Observations:

$$15 = q_0 - q_A - q_B + q_{AB}$$

$$45 = q_0 + q_A - q_B - q_{AB}$$

$$25 = q_0 - q_A + q_B - q_{AB}$$

$$75 = q_0 + q_A + q_B + q_{AB}$$

Solution:

$$y = 40 + 20x_A + 10x_B + 5x_A x_B$$

**Interpretation**: Mean performance = 40 MIPS

Effect of memory = 20 MIPS; Effect of cache = 10 MIPS

Interaction between memory and cache = 5 MIPS.

# Computation of Effects

| Experiment | A | B | y |
|---|---|---|---|
| 1 | -1 | -1 | $y_1$ |
| 2 | 1 | -1 | $y_2$ |
| 3 | -1 | 1 | $y_3$ |
| 4 | 1 | 1 | $y_4$ |

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

$$y_1 = q_0 - q_A - q_B + q_{AB}$$

$$y_2 = q_0 + q_A - q_B - q_{AB}$$

$$y_3 = q_0 - q_A + q_B - q_{AB}$$

$$y_4 = q_0 + q_A + q_B + q_{AB}$$

# Computation of Effects (Cont)

**Solution**:

$$q_0 = \frac{1}{4}(y_1 + y_2 + y_3 + y_4)$$

$$q_A = \frac{1}{4}(-y_1 + y_2 - y_3 + y_4)$$

$$q_B = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4)$$

$$q_{AB} = \frac{1}{4}(y_1 - y_2 - y_3 + y_4)$$

Notice that effects are linear combinations of responses.

Sum of the coefficients is zero ⇒ **contrasts**.

# Computation of Effects (Cont)

| Experiment | A | B | y |
|---|---|---|---|
| 1 | -1 | -1 | $y_1$ |
| 2 | 1 | -1 | $y_2$ |
| 3 | -1 | 1 | $y_3$ |
| 4 | 1 | 1 | $y_4$ |

$$q_A = \frac{1}{4}(-y_1 + y_2 - y_3 + y_4)$$

$$q_B = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4)$$

Notice:

$q_A$ = Column A × Column y

$q_B$ = Column B × Column y

# Sign Table Method

| I | A | B | AB | y |
|---|---|---|---|---|
| 1 | -1 | -1 | 1 | 15 |
| 1 | 1 | -1 | -1 | 45 |
| 1 | -1 | 1 | -1 | 25 |
| 1 | 1 | 1 | 1 | 75 |
| 160 | 80 | 40 | 20 | Total |
| 40 | 20 | 10 | 5 | Total/4 |

# Allocation of Variation

- Importance of a factor = proportion of the *variation* explained

$$\text{Sample Variance of } y = s_y^2 = \frac{\sum_{i=1}^{2^2}(y_i - \bar{y})^2}{2^2 - 1}$$

$$\text{Total Variation of } y = \text{SST} = \sum_{i=1}^{2^2}(y_i - \bar{y})^2$$

- For a $2^2$ design:

$$\text{SST} = 2^2 q_A^2 + 2^2 q_B^2 + 2^2 q_{AB}^2 = \text{SSA} + \text{SSB} + \text{SSAB}$$

- Variation due to A = SSA = $2^2 q_A^2$
- Variation due to B = SSB = $2^2 q_B^2$
- Variation due to interaction = SSAB = $2^2 q_{AB}^2$
- Fraction explained by A = $\frac{SSA}{SST}$

Variation ≠ Variance

# Derivation

- Model:

  $$y_i = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi}$$

  Notice

1. The sum of entries in each column is zero:

   $\sum_{i=1}^{4} x_{Ai} = 0; \sum_{i=1}^{4} x_{Bi} = 0; \sum_{i=1}^{4} x_{Ai} x_{Bi} = 0;$

2. The sum of the squares of entries in each column is 4:

$$\begin{aligned} \sum_{i=1}^{4} x_{Ai}^2 &= 4 \\ \sum_{i=1}^{4} x_{Bi}^2 &= 4 \\ \sum_{i=1}^{4} (x_{Ai} x_{Bi})^2 &= 4 \end{aligned}$$

# Derivation (Cont)

3. The columns are orthogonal (inner product of any two columns is zero):

$$\sum_{i=1}^{4} x_{Ai} x_{Bi} = 0$$

$$\sum_{i=1}^{4} x_{Ai} (x_{Ai} x_{Bi}) = 0$$

$$\sum_{i=1}^{4} x_{Bi} (x_{Ai} x_{Bi}) = 0$$

# Derivation (Cont)

- Sample mean $\bar{y}$

$$= \frac{1}{4}\sum_{i=1}^{4} y_i$$

$$= \frac{1}{4}\sum_{i=1}^{4} (q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi})$$

$$= \frac{1}{4}\sum_{i=1}^{4} q_0 + \frac{1}{4} q_A \sum_{i=1}^{4} x_{Ai}$$

$$+ q_B \frac{1}{4}\sum_{i=1}^{4} x_{Bi} + q_{AB} \frac{1}{4}\sum_{i=1}^{4} x_{Ai} x_{Bi}$$

$$= q_0$$

# Derivation (Cont)

- Variation of y

$$
\begin{aligned}
&= \sum_{i=1}^{4} (y_i - \bar{y})^2 \\
&= \sum_{i=1}^{4} (q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi})^2 \\
&= \sum_{i=1}^{4} (q_A x_{Ai})^2 + \sum_{i=1}^{4} (q_B x_{Bi})^2 \\
&+ \sum_{i=1}^{4} (q_{AB} x_{Ai} x_{Bi})^2 + \text{Product terms} \\
&= q_A^2 \sum_{i=1}^{4} (x_{Ai})^2 + q_B^2 \sum_{i=1}^{4} (x_{Bi})^2 \\
&+ q_{AB}^2 \sum_{i=1}^{4} (x_{Ai} x_{Bi})^2 + 0 \\
&= 4q_A^2 + 4q_B^2 + 4q_{AB}^2
\end{aligned}
$$

# Example 17.2

- Memory-cache study:

$$\bar{y} = \frac{1}{4}(15 + 55 + 25 + 75) = 40$$

$$\begin{aligned} \text{Total Variation} &= \sum_{i=1}^{4} (y_i - \bar{y})^2 \\ &= (25^2 + 15^2 + 15^2 + 35^2) \\ &= 2100 \\ &= 4 \times 20^2 + 4 \times 10^2 + 4 \times 5^2 \end{aligned}$$

- Total variation= 2100
  Variation due to Memory = 1600 (76%)
  Variation due to cache = 400 (19%)
  Variation due to interaction = 100 (5%)

# Case Study 17.1: Interconnection Nets

- Memory interconnection networks: Omega and Crossbar.
- Memory reference patterns: *Random* and *Matrix*
- Fixed factors:
  - Number of processors was fixed at 16.
  - Queued requests were not buffered but blocked.
  - Circuit switching instead of packet switching.
  - Random arbitration instead of round robin.
  - Infinite interleaving of memory ⇒ no memory bank contention.

# $2^2$ Design for Interconnection Networks

Factors Used in the Interconnection Network Study

| | | Level | |
|---|---|---|---|
| Symbol | Factor | -1 | 1 |
| A | Type of the network | Crossbar | Omega |
| B | Address Pattern Used | Random | Matrix |

| | | Response | | |
|---|---|---|---|---|
| A | B | Throughput T | 90% Transit N | Response R |
| -1 | -1 | 0.0641 | 3 | 1.655 |
| 1 | -1 | 0.4220 | 5 | 2.378 |
| -1 | 1 | 0.7922 | 2 | 1.262 |
| 1 | 1 | 0.4717 | 4 | 2.190 |

# Interconnection Networks Results

| Parameter | Mean Estimate | | | Variation Explained | | |
|---|---|---|---|---|---|---|
| | T | N | R | T | N | R |
| $q_0$ | 0.5725 | 3.5 | 1.871 | | | |
| $q_A$ | 0.0595 | -0.5 | -0.145 | 17.2% | 20% | 10.9% |
| $q_B$ | -0.1257 | 1.0 | 0.413 | 77.0% | 80% | 87.8% |
| $q_{AB}$ | -0.0346 | 0.0 | 0.051 | 5.8% | 0% | 1.3% |

- Average throughput = 0.5725
- Most effective factor = B = Reference pattern
  ⇒ The address patterns chosen are very different.
- Reference pattern explains ¨ 0.1257 (77%) of variation.
- Effect of network type = 0.0595
  Omega networks = Average + 0.0595
  Crossbar networks = Average - 0.0595
- Slight interaction (0.0346) between reference pattern and network type.

# General $2^k$ Factorial Designs

- k factors at two levels each.

  $2^k$ experiments.

  $2^k$ effects:

  k main effects

  $\binom{k}{2}$ two factor interactions

  $\binom{k}{3}$ three factor interactions...

# $2^k$ Design Example

- Three factors in designing a machine:
  - Cache size
  - Memory size
  - Number of processors

| | Factor | Level -1 | Level 1 |
|---|---|---|---|
| A | Memory Size | 4MB | 16MB |
| B | Cache Size | 1kB | 2kB |
| C | Number of Processors | 1 | 2 |

# $2^k$ Design Example (cont)

| Cache Size | 4M Bytes | | 16M Bytes | |
|---|---|---|---|---|
| | 1 Proc | 2 Proc | 1 Proc | 2 Proc |
| 1K Byte | 14 | 46 | 22 | 58 |
| 2K Byte | 10 | 50 | 34 | 86 |

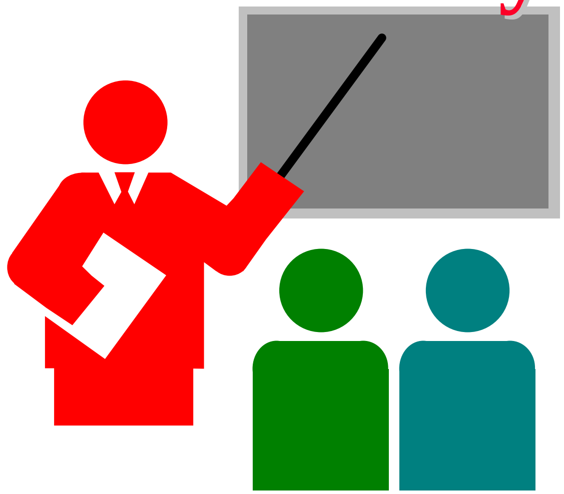
| I | A | B | C | AB | AC | BC | ABC | y |
|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 | 14 |
| 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 22 |
| 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 | 10 |
| 1 | 1 | 1 | -1 | 1 | -1 | -1 | -1 | 34 |
| 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 46 |
| 1 | 1 | -1 | 1 | -1 | 1 | -1 | -1 | 58 |
| 1 | -1 | 1 | 1 | -1 | -1 | 1 | -1 | 50 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 86 |
| 320 | 80 | 40 | 160 | 40 | 16 | 24 | 9 | Total |
| 40 | 10 | 5 | 20 | 5 | 2 | 3 | 1 | Total/8 |

# Analysis of $2^k$ Design

$$
\begin{aligned}
\text{SST} &= 2^3(q_A^2 + q_B^2 + q_C^2 + q_{AB}^2 + q_{AC}^2 + q_{BC}^2 + q_{ABC}^2) \\
&= 8(10^2 + 5^2 + 20^2 + 5^2 + 2^2 + 3^2 + 1^2) \\
&= 800 + 200 + 3200 + 200 + 32 + 72 + 8 = 4512 \\
&= 18\% + 4\% + 71\% + 4\% + 1\% + 2\% + 0\% \\
&= 100\%
\end{aligned}
$$

- Number of Processors (C) is the most important factor.

# Summary



- $2^k$ design allows k factors to be studied at two levels each
- Can compute main effects and all multi-factors interactions
- Easy computation using sign table method
- Easy allocation of variation using squares of effects

# Exercise 17.1

Analyze the $2^3$ design:

| | $A_1$ | | $A_2$ | |
|---|---|---|---|---|
| | $C_1$ | $C_2$ | $C_1$ | $C_2$ |
| $B_1$ | 100 | 15 | 120 | 10 |
| $B_2$ | 40 | 30 | 20 | 50 |

- Quantify main effects and all interactions.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.

# Homework

**Modified** Exercise 17.1 Analyze the $2^3$ design:

| | $A_1$ | | $A_2$ | |
|---|---|---|---|---|
| | $C_1$ | $C_2$ | $C_1$ | $C_2$ |
| $B_1$ | 110 | 15 | 120 | 10 |
| $B_2$ | 60 | 30 | 40 | 50 |

- Quantify main effects and all interactions.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.