Computer Systems Performance Analysis: Design of Experiments

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The Audio/Video recordings of this tuorial are available at:

http://www.cse.wustl.edu/~jain/tutorials/ied_tut.htm

Performance Analysis

- Performance = Measurement, Simulation, Analytical Modeling
- Both measurement and simulation require resources and time
- Performance is affected by many factors:
  - For example: Network appliance performance is affected by CPU, Disk, network card, packet sizes
- Each of these factors can have several levels:For example:
  - 3 types of CPUs: Single core, dual core, multicore
  - 4 types of disks: 4800 rpm, 5200 rpm, 7200 rpm, 10000 rpm
  - 2 types of network: 10 Mbps, 100 Mpbs, 1 Gbps, 10 Gbps
  - 6 packet sizes: 64B, 128KB, 512B, 1024B, 1518B, 9KB
- How many experiments do we need? $3\times4 \times 2 \times 6 = 144$
- What is the effect of CPU?

Experimental Design

- Design a proper set of experiments for measurement or simulation. Don't need to do all possible combinations.
- Develop a model that best describes the data obtained.
- Estimate the contribution of each factor to the performance.
- Isolate the measurement errors
- Estimate confidence intervals for model parameters.
- Check if the alternatives are significantly different.
- Check if the model is adequate.
- The techniques apply to all systems: Networks, Distributed Systems, Data bases, algorithms, …

Text Book

- R. Jain, "Art of Computer Systems Performance Analysis," Wiley, 1991, ISBN:0471503363 (Winner of the "1992 Best Computer Systems Book" Award from Computer Press Association")

1. Introduction to Design of Experiments
2. 2^k Factorial Designs
3. 2^kr Factorial Designs
4. 2^{k-p} Fractional Factorial Designs

Module 1: Introduction to Design of Experiments

- What is experimental design?
- Terminology
- Common mistakes
- Sample designs

Terminology

- **Factors**: Variables that affect the response variable.

  E.g., CPU type, memory size, number of disk drives, workload used, and user's educational level.

  Also called predictor variables or predictors.
- **Levels**: The values that a factor can assume, E.g., the CPU type has three levels: 68000, 8080, or Z80.

  # of disk drives has four levels.

  Also called **treatment**.
- **Replication**: Repetition of all or some experiments.
- **Design**: The number of experiments, the factor level and number of replications for each experiment.

  E.g., Full Factorial Design with 5 replications: $3 \times 3 \times 4 \times 3 \times 3$ or 324 experiments, each repeated five times.

Terminology (Cont)

- **Interaction** ⇒ Effect of one factor depends upon the level of the other.

Table 1: Noninteracting Factors

	A_1	A_2
B_1	3	5
B_2	6	8

Table 2: Interacting Factors

	A_1	A_2
B_1	3	5
B_2	6	9

Common Mistakes in Experimentation

- The variation due to experimental error is ignored.
- Important parameters are not controlled.
- Effects of different factors are not isolated
- Simple one-factor-at-a-time designs are used
- Interactions are ignored
- Too many experiments are conducted.
  Better: two phases.

Types of Experimental Designs

- **Simple Designs**: Vary one factor at a time

$$\#\text{ of Experiments} = 1 + \sum_{i=1}^{k} (n_i - 1)$$

  - Not statistically efficient.
  - Wrong conclusions if the factors have interaction.
  - Not recommended.
- **Full Factorial Design**: All combinations.

$$\#\text{ of Experiments} = \prod_{i=1}^{k} n_i$$

  - Can find the effect of all factors.
  - Too much time and money.
  - May try 2^k design first.

Types of Experimental Designs (Cont)

- Fractional Factorial Designs: Less than Full Factorial
  - Save time and expense.
  - Less information.
  - May not get all interactions.
  - Not a problem if negligible interactions

Example

Personal workstation design

1. Processor: 68000, Z80, or 8086.
2. Memory size: 512K, 2M, or 8M bytes
3. Number of Disks: One, two, three, or four
4. Workload: Secretarial, managerial, or scientific.
5. User education: High school, college, or post-graduate level.

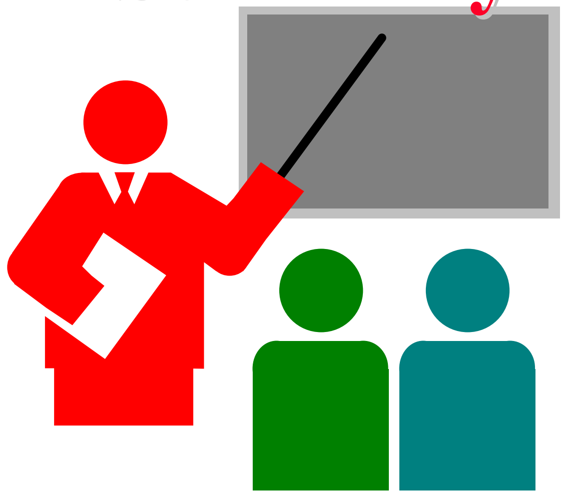
Five **Factors** at 3x3x4x3x3 **levels**

A Sample Fractional Factorial Design

- Workstation Design:
  (3 CPUs)(3 Memory levels)(3 workloads)(3 ed levels)
  = 81 experiments

Experiment Number	CPU	Memory Level	Workload Type	Educational Level
1	68000	512K	Managerial	High School
2	68000	2M	Scientific	Post-graduate
3	68000	8M	Secretarial	College
4	Z80	512K	Scientific	College
5	Z80	2M	Secretarial	High School
6	Z80	8M	Managerial	Post-graduate
7	8086	512K	Secretarial	Post-graduate
8	8086	2M	Managerial	College
9	8086	8M	Scientific	High School

Summary I



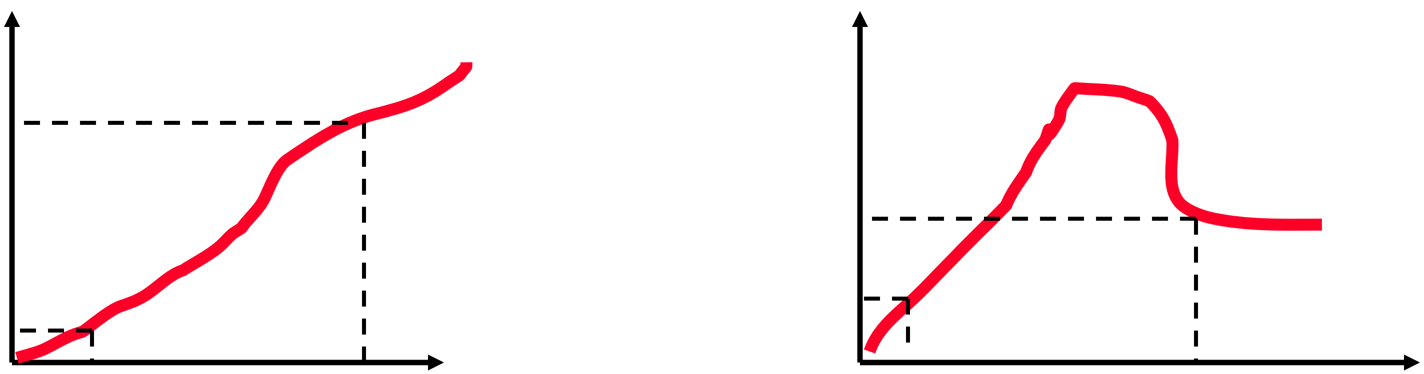
- Goal of proper experimental design is to get the maximum information with minimum number of experiments
- Factors, levels, full-factorial designs

Module 2:
2^k Factorial Designs

- 2^2 Factorial Designs
- Model
- Computation of Effects
- Sign Table Method
- Allocation of Variation
- General 2^k Factorial Designs

2^k Factorial Designs

- k factors, each at two levels.
- Easy to analyze.
- Helps in sorting out impact of factors.
- Good at the beginning of a study.
- Valid only if the effect is unidirectional.
  E.g., memory size, the number of disk drives



2^2 Factorial Designs

- Two factors, each at two levels.

Performance in MIPS

Cache Size	Memory Size	
	4M Bytes	16M Bytes
1K	15	45
2K	25	75

$$x_A = \begin{cases} -1 & \text{if 4M bytes memory} \\ 1 & \text{if 16M bytes memory} \end{cases}$$

$$x_B = \begin{cases} -1 & \text{if 1K bytes cache} \\ 1 & \text{if 2K bytes cache} \end{cases}$$

Model

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

Observations:

$$15 = q_0 - q_A - q_B + q_{AB}$$

$$45 = q_0 + q_A - q_B - q_{AB}$$

$$25 = q_0 - q_A + q_B - q_{AB}$$

$$75 = q_0 + q_A + q_B + q_{AB}$$

Solution:

$$y = 40 + 20x_A + 10x_B + 5x_A x_B$$

Interpretation: Mean performance = 40 MIPS

Effect of memory = 20 MIPS; Effect of cache = 10 MIPS

Interaction between memory and cache = 5 MIPS.

Sign Table Method

I	A	B	AB	y
1	-1	-1	1	15
1	1	-1	-1	45
1	-1	1	-1	25
1	1	1	1	75
160	80	40	20	Total
40	20	10	5	Total/4

Allocation of Variation

- Importance of a factor = proportion of the *variation* explained

$$\text{Sample Variance of } y = s_y^2 = \frac{\sum_{i=1}^{2^2}(y_i - \bar{y})^2}{2^2 - 1}$$

$$\text{Total Variation of } y = \text{SST} = \sum_{i=1}^{2^2}(y_i - \bar{y})^2$$

- For a 2^2 design:

$$\text{SST} = 2^2 q_A^2 + 2^2 q_B^2 + 2^2 q_{AB}^2 = \text{SSA} + \text{SSB} + \text{SSAB}$$

- Variation due to A = SSA = $2^2\, q_A{}^2$
- Variation due to B = SSB = $2^2\, q_B{}^2$
- Variation due to interaction = SSAB = $2^2\, q_{AB}{}^2$
- Fraction explained by A = $\frac{SSA}{SST}$

Variation ≠ Variance

Example 17.2

- Memory-cache study:

$$\bar{y} = \frac{1}{4}(15 + 55 + 25 + 75) = 40$$

$$\begin{aligned} \text{Total Variation} &= \sum_{i=1}^{4} (y_i - \bar{y})^2 \\ &= (25^2 + 15^2 + 15^2 + 35^2) \\ &= 2100 \\ &= 4 \times 20^2 + 4 \times 10^2 + 4 \times 5^2 \end{aligned}$$

- Total variation= 2100
  Variation due to Memory = 1600 (76%)
  Variation due to cache = 400 (19%)
  Variation due to interaction = 100 (5%)

Case Study 17.1: Interconnection Nets

- Memory interconnection networks: Omega and Crossbar.
- Memory reference patterns: *Random* and *Matrix*
- Fixed factors:
  - Number of processors was fixed at 16.
  - Queued requests were not buffered but blocked.
  - Circuit switching instead of packet switching.
  - Random arbitration instead of round robin.
  - Infinite interleaving of memory ⇒ no memory bank contention.

2^2 Design for Interconnection Networks

Factors Used in the Interconnection Network Study

Symbol	Factor	Level -1	Level 1
A	Type of the network	Crossbar	Omega
B	Address Pattern Used	Random	Matrix

A	B	Response: Throughput T	Response: 90% Transit N	Response: Response R
-1	-1	0.0641	3	1.655
1	-1	0.4220	5	2.378
-1	1	0.7922	2	1.262
1	1	0.4717	4	2.190

Interconnection Networks Results

Parameter	Mean Estimate			Variation Explained		
	T	N	R	T	N	R
q_0	0.5725	3.5	1.871			
q_A	0.0595	-0.5	-0.145	17.2%	20%	10.9%
q_B	-0.1257	1.0	0.413	77.0%	80%	87.8%
q_{AB}	-0.0346	0.0	0.051	5.8%	0%	1.3%

- Average throughput = 0.5725
- Most effective factor = B = Reference pattern
  ⇒ The address patterns chosen are very different.
- Reference pattern explains ∓ 0.1257 (77%) of variation.
- Effect of network type = 0.0595
  Omega networks = Average + 0.0595
  Crossbar networks = Average - 0.0595
- Slight interaction (0.0346) between reference pattern and network type.

General 2^k Factorial Designs

- k factors at two levels each.

  2^k experiments.

  2^k effects:

$$\begin{array}{l} \text{k main effects} \\ \left(\begin{array}{c} k \\ 2 \end{array}\right) \text{two factor interactions} \\ \left(\begin{array}{c} k \\ 3 \end{array}\right) \text{three factor interactions...} \end{array}$$

2^k Design Example

- Three factors in designing a machine:
  - Cache size
  - Memory size
  - Number of processors

	Factor	Level -1	Level 1
A	Memory Size	4MB	16MB
B	Cache Size	1kB	2kB
C	Number of Processors	1	2

$$y = q_0 + q_A x_A + q_B x_B + q_C x_C + q_{AB} x_A x_B + q_{AC} x_A x_C + q_{BC} x_B x_C + q_{ABC} x_A x_B x_C$$

2^k Design Example (cont)

Cache Size	4M Bytes		16M Bytes	
	1 Proc	2 Proc	1 Proc	2 Proc
1K Byte	14	46	22	58
2K Byte	10	50	34	86

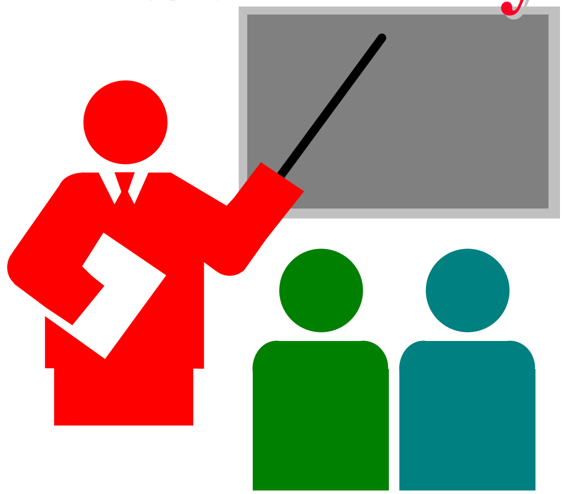
I	A	B	C	AB	AC	BC	ABC	y
1	-1	-1	-1	1	1	1	-1	14
1	1	-1	-1	-1	-1	1	1	22
1	-1	1	-1	-1	1	-1	1	10
1	1	1	-1	1	-1	-1	-1	34
1	-1	-1	1	1	-1	-1	1	46
1	1	-1	1	-1	1	-1	-1	58
1	-1	1	1	-1	-1	1	-1	50
1	1	1	1	1	1	1	1	86
320	80	40	160	40	16	24	9	Total
40	10	5	20	5	2	3	1	Total/8

Analysis of 2^k Design

$$\begin{aligned} \text{SST} &= 2^3(q_A^2 + q_B^2 + q_C^2 + q_{AB}^2 + q_{AC}^2 + q_{BC}^2 + q_{ABC}^2) \\ &= 8(10^2 + 5^2 + 20^2 + 5^2 + 2^2 + 3^2 + 1^2) \\ &= 800 + 200 + 3200 + 200 + 32 + 72 + 8 = 4512 \\ &= 18\% + 4\% + 71\% + 4\% + 1\% + 2\% + 0\% \\ &= 100\% \end{aligned}$$

- Number of Processors (C) is the most important factor.

Summary



- 2^k design allows k factors to be studied at two levels each
- Can compute main effects and all multi-factors interactions
- Easy computation using sign table method
- Easy allocation of variation using squares of effects

Module 3: $2^k r$ Factorial Designs

- Computation of Effects
- Estimation of Experimental Errors
- Allocation of Variation
- Confidence Intervals for Effects
- Confidence Intervals for Predicted Responses
- Visual Tests for Verifying the assumptions
- Multiplicative Models

$2^k r$ Factorial Designs

- r replications of 2^k Experiments
  ⇒ $2^k r$ observations.
  ⇒ Allows estimation of experimental errors.

- Model:

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e$$

- e = Experimental error

Computation of Effects

Simply use means of r measurements

I	A	B	A B	y	Mean $\bar{y}$
1	-1	-1	1	(15, 18, 12)	15
1	1	-1	-1	(45, 48, 51)	48
1	-1	1	-1	(25, 28, 19)	24
1	1	1	1	(75, 75, 81)	77
164	86	38	20		total
41	21.5	9.5	5		total/4

- Effects: q_0= 41, q_A= 21.5, q_B= 9.5, q_{AB}= 5.

Experimental Errors: Example

- Estimated Response:

$$\hat{y}_1 = q_0 - q_A - q_B + q_{AB} = 41 - 21.5 - 9.5 + 5 = 15$$

- Experimental errors:

$$e_{11} = y_{11} - \hat{y}_1 = 15 - 15 = 0$$

i	Effect I	Effect A	Effect B	Effect A B	Estimated Response	Measured Responses			Errors		
	41	21.5	9.5	5	$\hat{y}_i$	y_{i1}	y_{i2}	y_{i3}	e_{i1}	e_{i2}	e_{i3}
1	1	-1	-1	1	15	15	18	12	0	3	-3
2	1	1	-1	-1	48	45	48	51	-3	0	3
3	1	-1	1	-1	24	25	28	19	1	4	-5
4	1	1	1	1	77	75	75	81	-2	-2	4

Allocation of Variation

- Total variation or total sum of squares:

$$\text{SST} = \sum_{i,j}(y_{ij} - \bar{y}_{..})^2$$

$$y_{ij} = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi} + e_{ij}$$

$$\begin{array}{ccccccccc}
\sum_{i,j}(y_{ij} - \bar{y}_{..})^2 & = & 2^2 r q_A^2 & + & 2^2 r q_B^2 & + & 2^2 r q_{AB}^2 & + & \sum_{i,j} e_{ij}^2 \\
\text{SST} & = & \text{SSA} & + & \text{SSB} & + & \text{SSAB} & + & \text{SSE} \\
7032 & = & 5547 & + & 1083 & + & 300 & + & 102 \\
100\% & = & 78.88\% & + & 15.4\% & + & 4.27\% & + & 1.45\%
\end{array}$$

Confidence Intervals For Effects

- Effects are random variables.
- Errors $\sim$ N(0,σ_e) $\Rightarrow$ y $\sim$ N($\bar{y}_{..}$, σ_e)
- Variance of errors:

$$s_e^2 = \frac{1}{2^2(r-1)} \sum_{ij} e_{ij}^2 = \frac{\text{SSE}}{2^2(r-1)} \underline{\triangle} \text{MSE}$$

- Similarly,

$$s_{q_A} = s_{q_B} = s_{q_{AB}} = \frac{s_e}{\sqrt{2^2 r}}$$

- Confidence intervals (CI) for the effects:

$$q_i \mp t_{[1-\alpha/2;2^2(r-1)]} s_{q_i}$$

- CI does not include a zero $\Rightarrow$ significant

Example 18.4

- For Memory-cache study: Standard deviation of errors:

$$s_e = \sqrt{\frac{\text{SSE}}{2^2(r-1)}} = \sqrt{\frac{102}{8}} = \sqrt{12.75} = 3.57$$

- Standard deviation of effects:

$$s_{q_i} = s_e/\sqrt{(2^2 r)} = 3.57/\sqrt{12} = 1.03$$

- For 90% Confidence: $t_{[0.95,8]} = 1.86$

- Confidence intervals: $q_i \mp (1.86)(1.03) = q_i \mp 1.92$

$q_0 = (39.08, 42.91)$

$q_A = (19.58, 23.41)$

$q_B = (7.58, 11.41)$

$q_{AB} = (3.08, 6.91)$

- No zero crossing ⇒ All effects are significant.

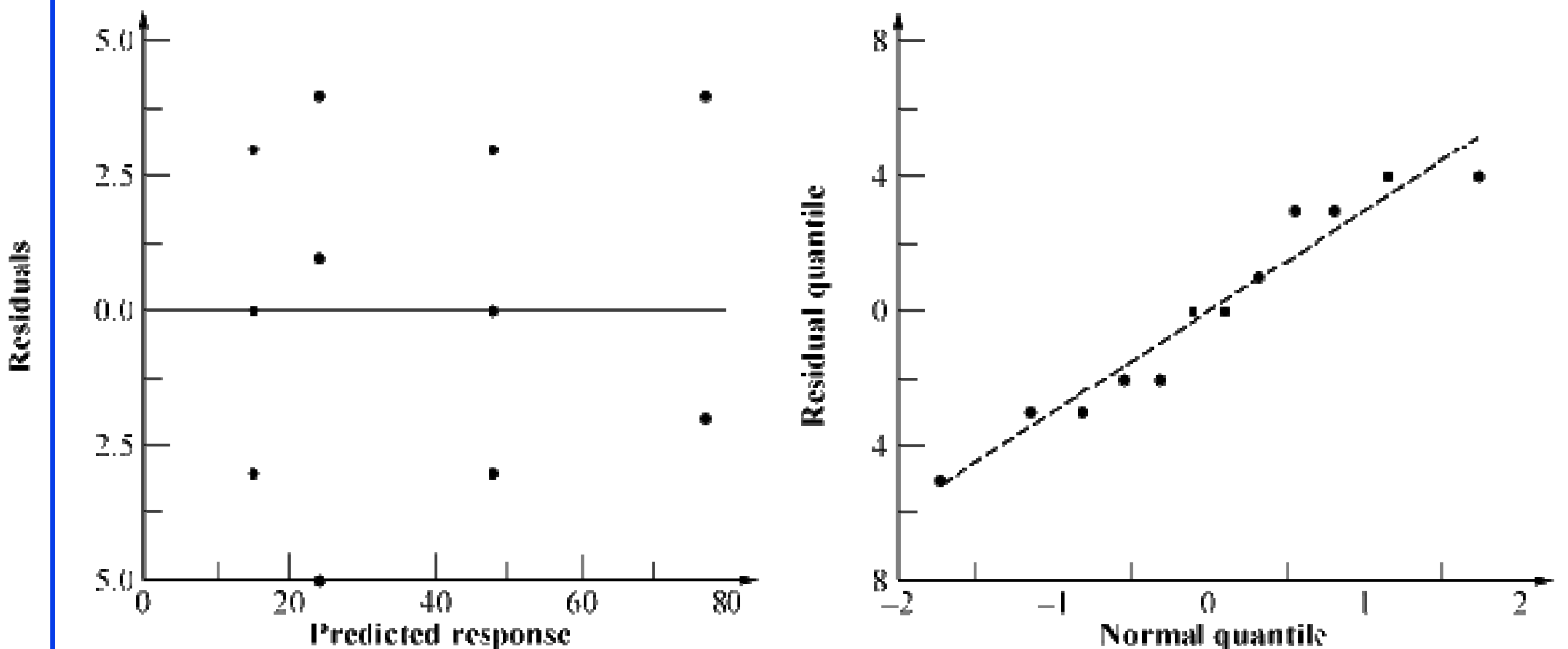
Assumptions

1. Errors are statistically independent.
2. Errors are additive.
3. Errors are normally distributed.
4. Errors have a constant standard deviation σ_e.
5. Effects of factors are additive
   $\Rightarrow$ observations are independent and normally distributed with constant variance.

Visual Tests

1. **Independent Errors**:
   - Scatter plot of residuals versus the predicted response $\hat{y}_i$
   - Magnitude of residuals < Magnitude of responses/10
     ⇒ Ignore trends
   - Plot the residuals as a function of the experiment number
   - Trend up or down ⇒ other factors or side effects
2. **Normally distributed errors**:
   Normal quantile-quantile plot of errors
3. **Constant Standard Deviation of Errors**:
   Scatter plot of y for various levels of the factor
   Spread at one level significantly different than that at other
   ⇒ Need transformation

Example 18.7: Memory-cache



Multiplicative Models

- Additive model:

  $$y_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij}$$

- Not valid if effects do not add.
  E.g., execution time of workloads.

  *i*th processor speed= v_i instructions/second.

  *j*th workload Size= w_j instructions

- The two effects multiply. Logarithm ⇒ additive model:

  $$\text{Execution Time } y_{ij} = v_i \times w_j$$

  $$\log(y_{ij}) = \log(v_i) + \log(w_j)$$

- Correct Model:

  $$y'_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij}$$

  Where, $y'_{ij}=\log(y_{ij})$

Multiplicative Model (Cont)

- Taking an antilog of effects:

  $u_A = 10^{q_A}$, $u_B = 10^{q_B}$, and $u_{AB} = 10^{q_{AB}}$

- u_A = ratio of MIPS rating of the two processors
- u_B = ratio of the size of the two workloads.
- Antilog of additive mean $q_0 \Rightarrow$ geometric mean

$$\dot{y} = 10^{q_0} = (y_1 y_2 \cdots y_n)^{1/n} \quad n = 2^2 r$$

Example 18.8: Execution Times

Analysis Using an Additive Model

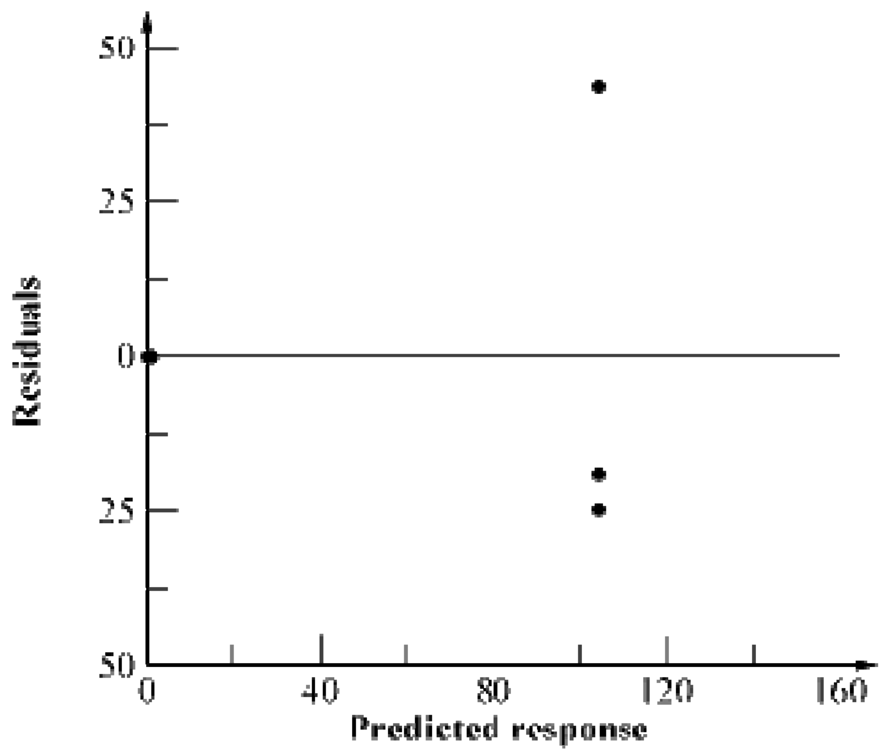
I	A	B	AB	y	Mean $\bar{y}$
1	-1	-1	1	(85.10, 79.50, 147.90)	104.170
1	1	-1	-1	(0.891, 1.047, 1.072)	1.003
1	-1	1	-1	(0.955, 0.933, 1.122)	1.003
1	1	1	1	(0.0148, 0.0126, 0.0118)	0.013
106.19	-104.15	-104.15	102.17	total	
26.55	-26.04	-26.04	25.54	total/4	

Additive model is not valid because:

- Physical consideration ⇒ effects of workload and processors do not add. They multiply.
- Large range for y. y_{max}/y_{min}= 147.90/0.0118 or 12,534 ⇒ log transformation
- Taking an arithmetic mean of 114.17 and 0.013 is inappropriate.

Example 18.8 (Cont)

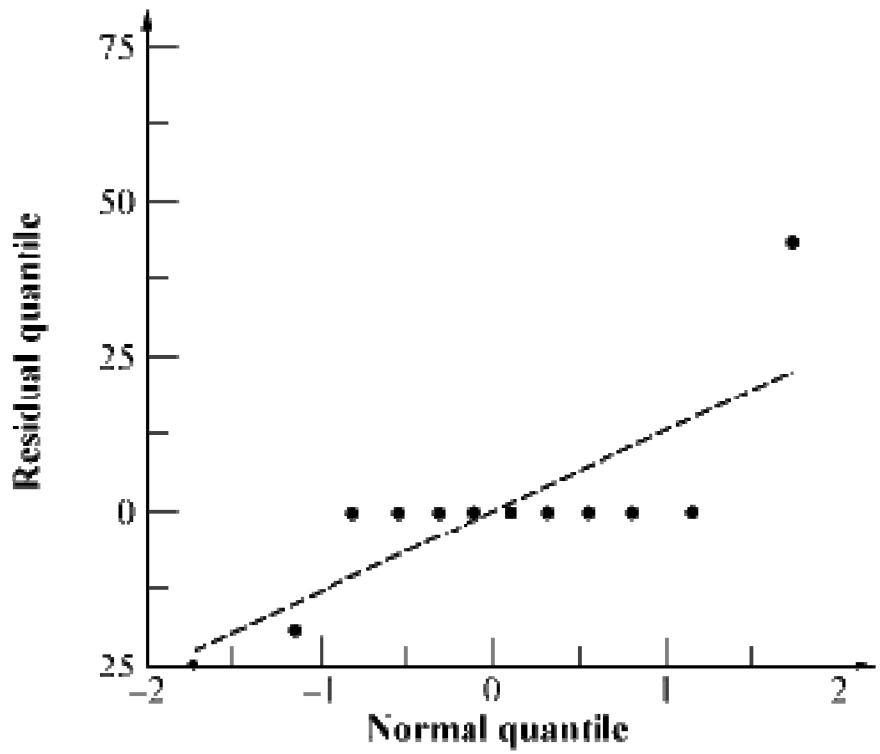
- The residuals are not small as compared to the response.



- The spread of residuals is large at larger value of the response.
  ⇒ log transformation

Example 18.8 (Cont)

- Residual distribution has a longer tail than normal



Analysis Using Multiplicative Model

Data After Log Transformation

I	A	B	AB	y	Mean $\bar{y}$
1	-1	-1	1	(1.93, 1.90, 2.17)	2.00
1	1	-1	-1	(-0.05, 0.02, 0.03)	0.00
1	-1	1	-1	(-0.02, -0.03, 0.05)	0.00
1	1	1	1	(-1.83, -1.90, -1.93)	-1.89
0.11	-3.89	-3.89	0.11	total	
0.03	-0.97	-0.97	0.03	total/4	

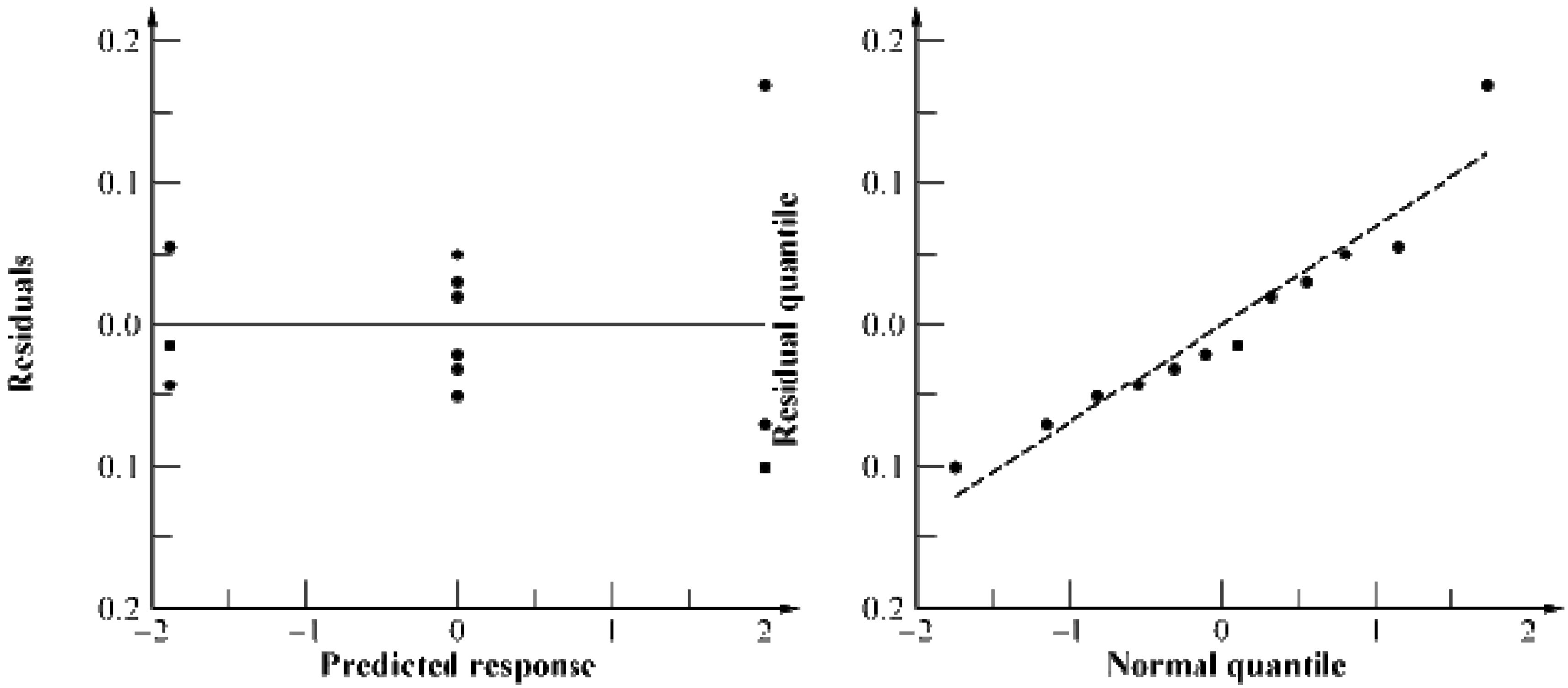
Variation Explained by the Two Models

	Additive Model			Multiplicative Model		
Factor	Effect	% Var.	Conf. Interval	Effect	% Var.	Conf. Interval
I	26.55		(16.35, 36.74)	0.03		(-0.02, 0.07)†
A	-26.04	30.1%	(-36.23, -15.84)	-0.97	49.9%	(-1.02, -0.93)
B	-26.04	30.1%	(-36.23, -15.84)	-0.97	49.9%	(-1.02, -0.93)
AB	25.54	29.0%	(15.35, 35.74)	0.03	0.0%	(-0.02, 0.07)†
e		10.8%			0.2%	

† $\Rightarrow$ Not Significant

- With multiplicative model:
  - Interaction is almost zero.
  - Unexplained variation is only 0.2%

Visual Tests



- **Conclusion**: Multiplicative model is better than the additive model.

Interpretation of Results

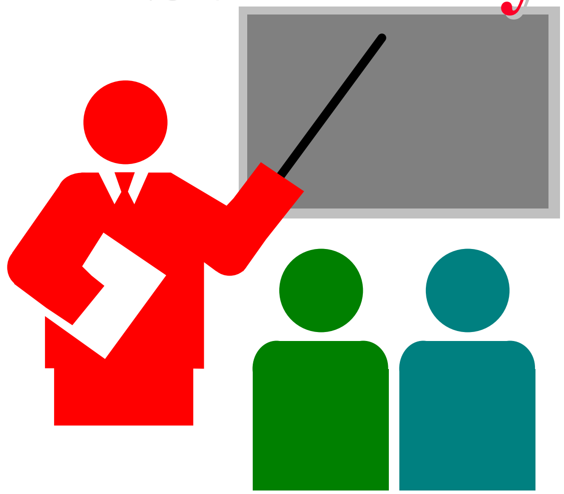
$$\log(y) = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e$$

$$\begin{aligned} \Rightarrow y &= 10^{q_0} 10^{q_A x_A} 10^{q_B x_B} 10^{q_{AB} x_A x_B} 10^e \\ &= 10^{0.03} 10^{-0.97 x_A} 10^{-0.97 x_B} 10^{0.03 x_A x_B} 10^e \\ &= 1.07 \times 0.107^{x_A} \times 0.107^{x_B} \times 1.07^{x_A x_B} 10^e \end{aligned}$$

- The time for an average processor on an average benchmark is 1.07.
- The time on processor A_1 is nine times (0.107^{-1}) that on an average processor. The time on A_2 is one ninth (0.107^1) of that on an average processor.
- MIPS rate for A_2 is 81 times that of A_1.
- Benchmark B_1 executes 81 times more instructions than B_2.
- The interaction is negligible.

  $\Rightarrow$ Results apply to all benchmarks and processors.

Summary



- Replications allow estimation of measurement errors
  $\Rightarrow$ Confidence Intervals of parameters
  Allocation of variation is proportional to square of effects
- Multiplicative models are appropriate if the factors multiply
- Visual tests for independence normal errors

Module 4: 2^{k-p} Fractional Factorial Designs

- 2^{k-p} Fractional Factorial Designs
- Sign Table for a 2^{k-p} Design
- Confounding
- Other Fractional Factorial Designs
- Algebra of Confounding
- Design Resolution

2^{k-p} Fractional Factorial Designs

- Large number of factors
  - ⇒ large number of experiments
  - ⇒ full factorial design too expensive
  - ⇒ Use a fractional factorial design
- 2^{k-p} design allows analyzing k factors with only 2^{k-p} experiments.

  2^{k-1} design requires only half as many experiments

  2^{k-2} design requires only one quarter of the experiments

Example: 2^{7-4} Design

Expt No.	A	B	C	D	E	F	G
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

- Study 7 factors with only 8 experiments!

Fractional Design Features

- Full factorial design is easy to analyze due to orthogonality of sign vectors.

  Fractional factorial designs also use orthogonal vectors. That is:

  - The sum of each column is zero.

$$\sum_i x_{ij} = 0 \quad \forall j$$

*j*th variable, *i*th experiment.

  - The sum of the products of any two columns is zero.

$$\sum_i x_{ij} x_{il} = 0 \quad \forall j \neq l$$

  - The sum of the squares of each column is 2^{7-4}, that is, 8.

$$\sum_i x_{ij}^2 = 8 \quad \forall j$$

Analysis of Fractional Factorial Designs

I	A	B	C	D	E	F	G	y
1	-1	-1	-1	1	1	1	-1	20
1	1	-1	-1	-1	-1	1	1	35
1	-1	1	-1	-1	1	-1	1	7
1	1	1	-1	1	-1	-1	-1	42
1	-1	-1	1	1	-1	-1	1	36
1	1	-1	1	-1	1	-1	-1	50
1	-1	1	1	-1	-1	1	-1	45
1	1	1	1	1	1	1	1	82
317	101	35	109	43	1	47	3	Total
39.62	12.62	4.37	13.62	5.37	0.125	5.87	0.37	Total/8

- Factors A through G explain 37.26%, 4.74%, 43.40%, 6.75%, 0%, 8.06%, and 0.03% of variation, respectively.

  ⇒ Use only factors C and A for further experimentation.

Sign Table for a 2^{k-p} Design

Steps:

1. Prepare a sign table for a full factorial design with k-p factors.
2. Mark the first column I.
3. Mark the next k-p columns with the k-p factors.
4. Of the $(2^{k-p}-k-p-1)$ columns on the right, choose p columns and mark them with the p factors which were not chosen in step 1.

Example: 2^{7-4} Design

Expt No.	A	B	C	AB	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

Example: 2^{4-1} Design

Expt No.	A	B	C	AB	AC	BC	D
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

Confounding

- **Confounding**: Only the combined influence of two or more effects can be computed.

$$
\begin{aligned}
q_A &= \sum_i y_i x_{Ai} \\
&= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}
\end{aligned}
$$

$$
\begin{aligned}
q_D &= \sum_i y_i x_{Di} \\
&= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}
\end{aligned}
$$

Confounding (Cont)

$$q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci}$$
$$= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

$$q_D = q_{ABC}$$

$$q_D + q_{ABC} = \sum_i y_i x_{Ai} x_{Bi} x_{Ci}$$
$$= \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8}$$

- ⇒ Effects of D and ABC are confounded. Not a problem if q_{ABC} is negligible.

Confounding (Cont)

- Confounding representation: *D=ABC*

  Other Confoundings:

$$
\begin{aligned}
q_A &= q_{BCD} = \sum_i y_i x_{Ai} \\
&= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}
\end{aligned}
$$

$$\Rightarrow A = BCD$$

A=BCD, B=ACD, C=ABD, AB=CD, AC=BD, BC=AD, ABC=D, and I=ABCD

- *I=ABCD* ⇒ confounding of ABCD with the mean.

Other Fractional Factorial Designs

- A fractional factorial design is not unique. 2^p different designs.

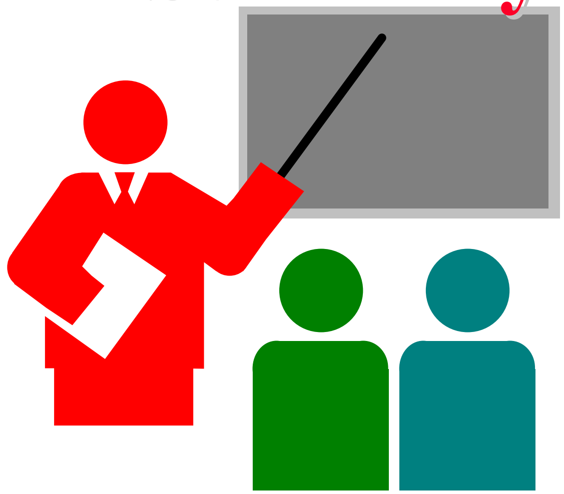
Another 2^{4-1} Experimental Design

Expt No.	A	B	C	D	AC	BC	ABC
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

- Confoundings: $I=ABD$, $A=BD$, $B=AD$, $C=ABCD$, $D=AB$, $AC=BCD$, $BC=ACD$, $ABC=CD$

Not as good as the previous design.

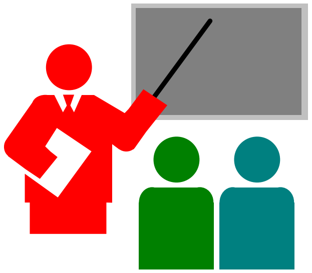
Summary



- Fractional factorial designs allow a large number of variables to be analyzed with a small number of experiments
- Many effects and interactions are confounded

Other Designs

- One factor with many levels
  e.g., 1 factor with 5 levels
- Two factors with different levels,
  e.g., 2 factors with 4×5 levels
- Multiple factors with different levels,
  e.g., 4 factors with 3×4×5×2 levels
- All these designs and others are discussed in the book.



Overall Summary

- 2^k design allows k factors to be studied at two levels each
- Can compute main effects and all multi-factors interactions
- Easy computation using sign table method
- Easy allocation of variation using squares of effects
- $2^k r$ design with replications allow estimation of measurement errors ⇒ Confidence Intervals of parameters
- Multiplicative models are appropriate if the factors multiply
- Visual tests for independence normal errors
- 2^{k-p} Fractional factorial designs allow a large number of variables to be analyzed with a small number of experiments
- Many effects and interactions are confounded