Practice Final Exam 2010

Please write your name at the top of this page. There are no *trick* questions on this exam, but approach each problem with an open mind. Please keep track of the time, and good luck.

1. (Very Short answer)

(a) Let \( A \) be the set \{1, 2, 3\}, and let \( R \) be the relation that includes exactly the ordered pairs: \{(1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (3,3)\};

i. List the elements that must be added to \( A \) to create the transitive closure of \( A \).

[Answer: (3,2),(2,3)]

(b) Consider the possible directed graphs \( G \) with 4 nodes.

i. (T or F) It is possible for \( G \) to have 23 edges? [Answer: No. The maximum number of edges for a directed graph with 4 nodes is 16]

ii. (T or F) It is possible for a node of \( G \) to be touched by no edges at all. [Answer: Yes. The graph even have 4 nodes and no edges at all...]

iii. How many possible graphs \( G \) are there? [Answer: For every pair of nodes \((x,y)\), including \( x \) the same as \( y \), the graph \( G \) could have an edge or not. so the number of possible graphs is \( 2^{16} \).]

(c) Let \( K_n \) denote the complete, undirected, graph with \( n \) nodes (i.e. every pair of the \( n \)-nodes contains an edge between them).

i. Draw \( K_5 \) [Answer: see figure below]

ii. When (for what values of \( n \)) does \( K_n \) have an Euler path? [Answer: When there are \( n \) nodes, every node has \( n-1 \) edges (because it has an edge to every other node in the graph. So when \( n \) is odd, every node has even degree. \( K_n \) is always connected (because it has every possible edge), and if the graph is connected and has all nodes with even degree it always has an Euler Tour. (It actually has an Euler cycle in this case, but that is also a tour).]

iii. When (for what values of \( n \)) does \( K_n \) have a Hamiltonian path? [Answer: \( K_n \) has a Hamiltonian path for all values of \( n \). In fact, you could create any order that you’d like to visit the nodes in, and the path exists for exactly that order, (because every possible edge exists in this graph).]

(d) (T or F) It is possible to have a graph that has a Hamiltonian Path but no Euler Path. [Answer: True: The complete graph with 4 nodes is an example of such a graph]

(e) (T or F) A Finite State Machine with 5 states can accept more than 6 different strings. [Answer: True: often, finite state machines accept infinite numbers of strings, such as the ones from our last homework that accepted any odd-length string that ended in a one]

(f) Define the term “Simple Path” as it relates to a graph. [Answer: A simple path is a path that never re-visits a node. A Hamiltonian path, for example, is a simple path that visits every node exactly once — but you do not have to visit all nodes to be a simple path.]

(g) Define the term “Degree of a node” as it relates to an undirected graph. [Answer: The degree of a node is the number of edges that are incident on that node — the number of edges that “touch” that node.]
2. Your honest friend comes to visit. You deal them a 5 card poker hand. What is the probability that it has exactly 3 cards of the same rank and 2 cards of different ranks? (for example, 3 queens, a 7 and a 4)?

[Answer: Number of hands that have this property: \( \binom{13}{1} \binom{12}{1} \binom{4}{3} 4^2 \), because, in order to construct such a hand you need to pick the rank of the 3 of a kind, then you need to pick the other 2 cards. Then for the 3 of a kind, you need to pick the suits of each card (so you have to choose 3 of the 4 suits). The remaining two cards can be any combination of the remaining suits. So the probability works out to: \( \frac{\binom{13}{1} \binom{12}{1} \binom{4}{3} 4^2}{52^5} \). That is the appropriate answer for an exam, but if you were to calculate it, it would become: 0.0211, or 2.1% which is quite a bit more likely that the flushes or straights we calculated earlier. ]

You start dealing to your honest friend, and she says after the first 2 cards, “oh cool, I have 2 sevens!”. What is the probability that after you deal all 5 cards to her, she will have exactly 3 cards of the same rank and 2 cards of different ranks?

[Answer: If your friend already has 2 sevens, then she needs to get another 7 (if she got 3 of a different rank, then she would only have the other rank and the 7s, not 3 of a kind and two other kinds). So the number of possibilities is \( 2 \binom{12}{2} 4^2 \), because she needs to choose one of the remaining 2 sevens, and then 2 remaining ranks from the 12 ranks that aren’t 7, and then the suits for each of those two cards cards. To compute a probability, we also need to know how many different possible hands can she get? After she has 2 sevens, there are 50 cards left in the deck, and she get 3 of them, so the number of possible hands is \( \binom{50}{3} \). Thus, the overall probability is: \( \frac{2 \binom{12}{2} 4^2}{\binom{50}{3}} \). This is a perfect answer for a test, but if you want to know the number, it works out to 0.1078, or about an 11% chance that she ends up with 3 of a kind given that she starts with 2 cards of the same rank. ]

3. The degree of a node in a undirected graph is the number of edges that touch that node. The degree list is a list of the degrees of all the nodes in a graph. Is it possible to have a graph with degree list (3,1,1)? Why or why not?

[Answer: You cannot have this set of degrees. (Try to draw it?). In particular, a node with degree of 3 has 3 edges leaving it. They have to go somewhere. But there are only 2 other nodes, and each of those nodes only has degree of 1, so there are only “endpoints” for 2 of the edges. Since you can’t have a dangling edge, this graph is not possible. ]

4. Suppose an undirected graph has exactly 2 nodes of odd degree. Let’s call those nodes \( u, v \). Prove there exists a path from \( u \) to \( v \).

[Answer: Suppose, for contradiction, that there is no path from \( u \) to \( v \). Then the graph has to have 2 completely separate components which contain \( u \) and \( v \). (if they weren’t separate, there would be some path from \( u \) to \( v \)). Now, let \( G’ \) be the graph that consists of all the nodes connect to \( u \) and all their edges. In \( G’ \), \( u \) still has odd degree (because it starts with odd degree in the original graph and still has all its edges), and all the other nodes in \( G’ \) have even degree. This is a contradiction.

Why? \( G’ \) cannot exist, because no graph can have only 1 node with odd degree. Why is this? Consider the sum of the degree of every node in \( G’ \). This counts every edge in \( G’ \) twice (because every edge adds to the degree of two nodes). So the sum of the degrees of the nodes in \( G’ \) is twice the number of edges in \( G’ \). Thus, the sum of the degrees of the nodes in \( G’ \) is even, but also, the sum of the degrees of the nodes in \( G’ \) is odd — because \( u \) is the only node
with odd degree. Therefore, under the assumption that \( u, v \) are in different components, we have a contradiction. Therefore there must be a path from \( u \) to \( v \).

5. Construct a finite state machine that accepts all binary strings that have at least 3 0's or an even number of 1's (or both). \([Answer: \text{See figure.}]\)