Practice Problems:
1. Prove for integers \( n \geq 0, 3|n^3 + 2n \)

2. Prove Bernoulli’s inequality:
   \[(1 + x)^n \geq 1 + nx\]
   for all non-negative integers \( n \), when \( x \) is a real number greater than -1.

3. Prove that \( \log_{\sqrt{5}}16 \) is irrational.

Problems to turn in:
1. For all integers \( k \), prove that if \( k^2 \) is a multiple of 3 then \( k \) is a multiple of 3.

   We prove the contrapositive: If \( k \) is not a multiple of 3 then \( k^2 \) is not a multiple of 3.

   If \( k \) is not a multiple of 3, then there is some integer \( m \) such that we can write \( k = 3m + 1 \) or \( k = 3m + 2 \).

   Case 1: \( k = 3m + 1 \)
   \[ k^2 = (3m + 1)^2 \]
   \[ = 9m^2 + 6m + 1 \]
   \[ = 3(3m^2 + 2m) + 1 \]

   so \( k^2 = 3(integer) + 1 \), so \( k^2 \) is not a multiple of 3.

   Case 2: \( k = 3m + 2 \)
   \[ k^2 = (3m + 2)^2 \]
   \[ = 9m^2 + 12m + 4 \]
   \[ = 3(3m^2 + 2m + 1) + 1 \]

   so \( k^2 = 3(integer) + 1 \), so \( k^2 \) is not a multiple of 3.

   Since both cases show that \( k^2 \) is not a multiple of 3, and since those are the only possibilities for \( k \) not itself a multiple of 3, we have shown:

   If \( k \) is not a multiple of 3 then \( k^2 \) is not a multiple of 3.

   and therefore also have shown:

   if \( k^2 \) is a multiple of 3 then \( k \) is a multiple of 3.

2. Prove that \( \sqrt{3} \) is irrational using a method similar to the one we used in class (about \( \sqrt{2} \)). (You may use the property from the previous question, whether or not you successfully proved it).

   Suppose, for contradiction, that \( \sqrt{3} \) is rational.

   Then, \( \sqrt{3} = \frac{a}{b} \) for integers \( a, b \) that are relatively prime.

   Then, \( 3 = \frac{a^2}{b^2} \).

   Then, \( 3b^2 = a^2 \).

   So \( a^2 \) is a multiple of 3

   So \( a \) is a multiple of 3 by problem 1.

   So \( a = 3k \) for some \( k \).

   So \( 3b^2 = (3k)^2 \)
So $3b^2 = 9k^2$
So $b^2 = k^2$
So $b^2$ is a multiple of 3.
So $b$ is a multiple of 3.
So $a, b$ are both multiples of 3, which contradicts that $a, b$ are relatively prime.

3. Prove, by induction on $n$, that the following equation holds (as long as $a$ is not exactly 1):

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

Base case: $n = 0$ (Could also use $n = 1$, it is my fault that I didn’t specify a starting condition).

$P(n)$: $\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$

$P(0)$: $\sum_{i=0}^{0} a^i = \frac{a^{0+1} - 1}{a - 1}$

$$\sum_{i=0}^{0} a^i = a^0 = 1 = \frac{a-1}{a-1}$$

Induction Hypothesis Assume $P(k)$: $\sum_{i=0}^{k} a^i = \frac{a^{k+1} - 1}{a - 1}$

Prove $P(k+1)$: $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2} - 1}{a - 1}$

$$\sum_{i=0}^{k+1} a^i = \sum_{i=0}^{k} a^i + a^{k+1} = \frac{a^{k+1} - 1}{a - 1} + a^{k+1}, \text{ By the induction hypothesis}$$

$$= \frac{a^{k+1} - 1 + (a-1)a^{k+1}}{a - 1}$$

$$= \frac{a^{k+2} - 1}{a - 1}$$

$$= \frac{a^{k+2} - 1 + a^{k+1} - a^{k+1}}{a - 1}$$

$$= \frac{a^{k+2} - 1}{a - 1}$$

$$= \frac{a^{k+2} - 1}{a - 1}$$