CSE 546: Computational Geometry

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Teaching

• Mostly on the board.
  - ... once I'm back on my feet
• Really would like you to read the book chapters ahead of time.
• Honestly, this is my favorite class to teach.
  - Not my area.
  - Interesting open problems
  - New problem domains of mobile and sensor networks, GIS, high-dimensional machine learning
Class

- [> 50%] Midterm + Final
- [~20%] ~5-6 Homeworks
  - (Problem sets)
  - Prove relevant geometric properties
  - Design Algorithms
- [10-30%] Final Project
• Geometric properties
  - # of points, edges, and triangles in triangulation?
  - If you randomly put points down on the plane, how many are on the convex boundary?
  - Planar graph colors?
• Problem decompositions
  - Spatial Divide and Conquer
  - Plane sweep
• Data structures
  - K-D trees, Voronoi Diagrams
4 key peripheral ideas

- **Randomized algorithms**
  - “Defeat the adversary through randomization”
    - (many algorithms like quicksort)
- **Amortized analysis**
  - “Occasional bad cases may not be so bad?”
- **Output sensitive algorithms**
  - “things you do once-per-result should be counted differently from the rest of the algorithm?”
    - (i.e., list all intersections of these $n$ line segments)
- **“General Position”**
  - “At first, assume the data isn’t pathological”.
    - (if you are given $n$ points, you can assume no 2 are at exactly the same location, no 3 are on exactly the same line, etc.)
I used to give big overview

... and talk about robot path planning, and protein folding, computer graphics and GIS applications... but it isn't my field. So instead today we will start with an example of why I like this field...

• Convex Hulls...
  - boundary of a set of 2D points.
Definition of convex hull

- **Pictorial:**
  - $S$

- **Mathematical:**

  *Def'n: Convex Hull [Valentine, 1964]*

The convex hull of a set $S$ is the intersection of all convex sets containing $S$. 
“convex”

Def’n: Convex set

A set $S$ is said to be convex if and only if the line segment joining every pair of points lies entirely within $S$. 
Convex Hull

- A point is on the convex hull if it cannot be expressed as the convex combination of any other points.
Why?

- (math reason) analogy to "extreme" points of a 1-D point set.
- (application reason) easy first cut for object intersection
“Easy Geometry”

Orientation test

• An ordered triple of points has an “orientation” property.
• $\text{orient}(a,b,c) > 0$ if “$c$ is left of the line from $a$ to $b$”
• $\text{orient}(a,b,c) < 0$ if “$c$ is right of the line from $a$ to $b$”
Uses of the orientation test

• Intersect(ab, cd)

• Inside(q, abc)

Orient(a, b, c) \times orient(a, b, d) < 0 \text{ AND } Orient(c, d, a) \times orient(c, d, b) < 0

Orient(abq) = orient(bcq) = orient(caq)

Orient(abq) > 0 \text{ and } orient(bcq) > 0 \text{ and } orient(caq) > 0
A side note on geometries:

• Invariant to scale, coordinate system center, and affine distortions.

• If your algorithm uses just orientation tests, your results are *invariant* to these distortions.
Brute Force

**Basic Idea**

\[
\text{HULL}(S) = \text{HULL}(E)
\]

\[\{P_1, P_2, \ldots, P_6\} = E \subseteq S, \text{ where } E = \text{the set of all extreme points}\]

**Algorithm**

1. Identify all extreme points?
2. Order these points so that they form a convex polygon. \(\leftarrow\) Sorting

**Theorem** [Hadwiger, 1964]

A point \(u\) is not an extreme point if \(u\) satisfies one of the following:

1. \(u\) lies in a triangle consisting of three points in \(S\).
2. \(u\) is not a vertex of a triangle.

Why (2) needed?

\[\binom{n}{3} = O(n^3)\] triangles

For each triangle, \(O(n)\) tests

\[\therefore O(n^4)\]

Other ways to identify extreme points?