Alpha Hulls, Shapes, and Weighted things.

How small changes in definitions change things?

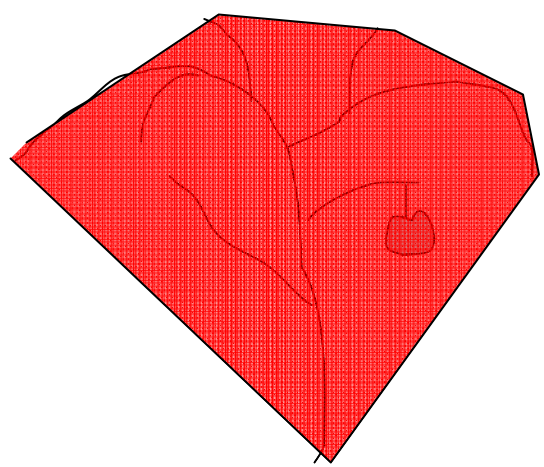
Convex Hull

- Additive Definition:
  - The shape given by the convex closure of the set of points.
    - (add all points on the line between any two points of the point set). Iterate until you cannot add any more points.
- Subtractive Definition
  - Take away all empty half-spaces

Convex Hull

- What is it good for?
  - It is the bounding regions of a point set
  - Used to speed up collision detection

- Not so good for describing shapes.



Convex Hull

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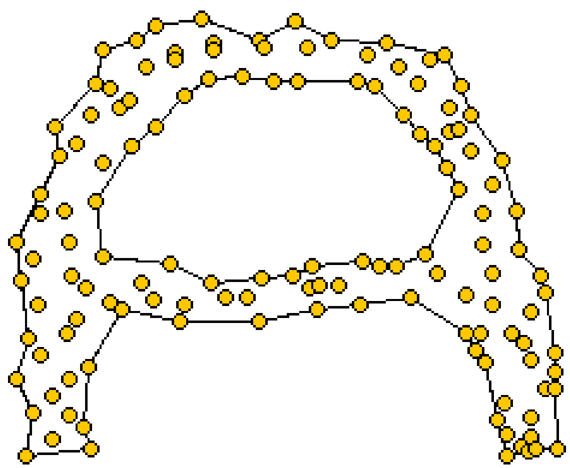
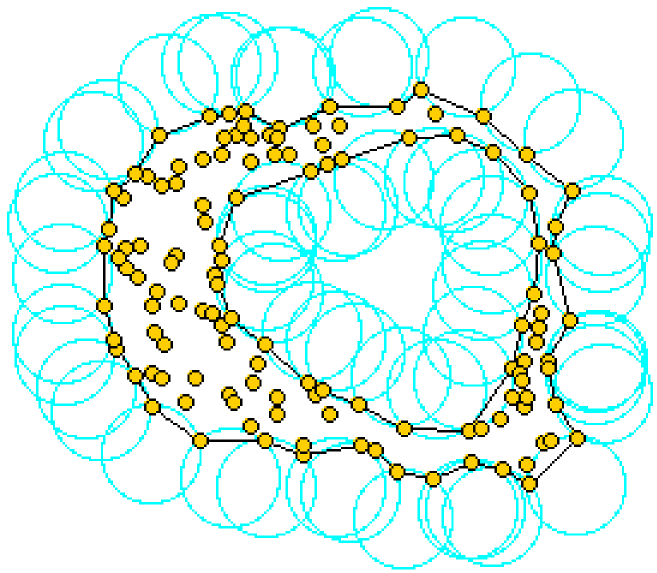
To get alpha shape, remove all empty spheres of radius alpha. Add straight edge between any two points that the sphere touches (so you don't get curved edges).

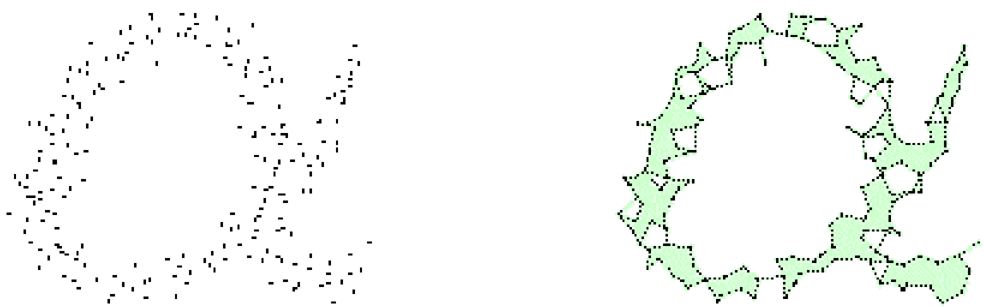
Alpha Shapes



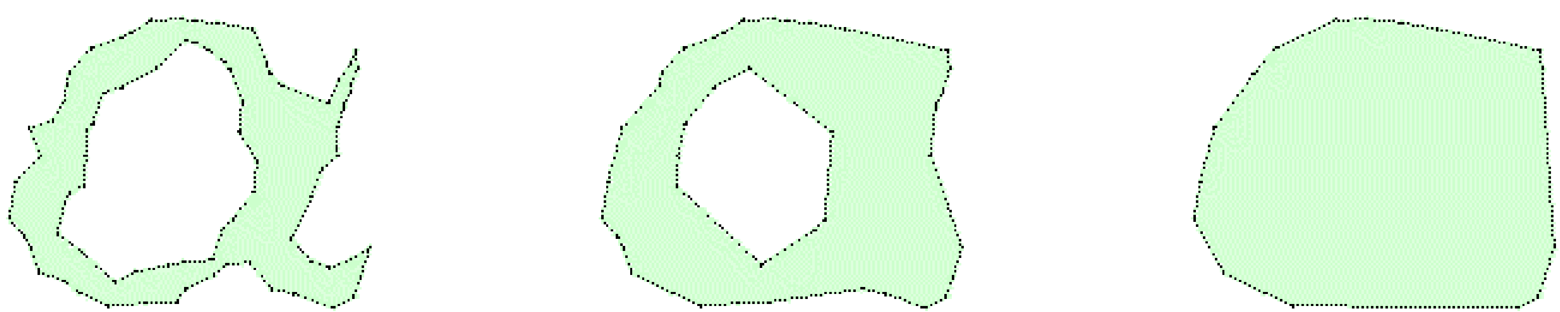


The space generated by point pairs that can be touched by an empty disc of radius alpha.

Alpha Shapes

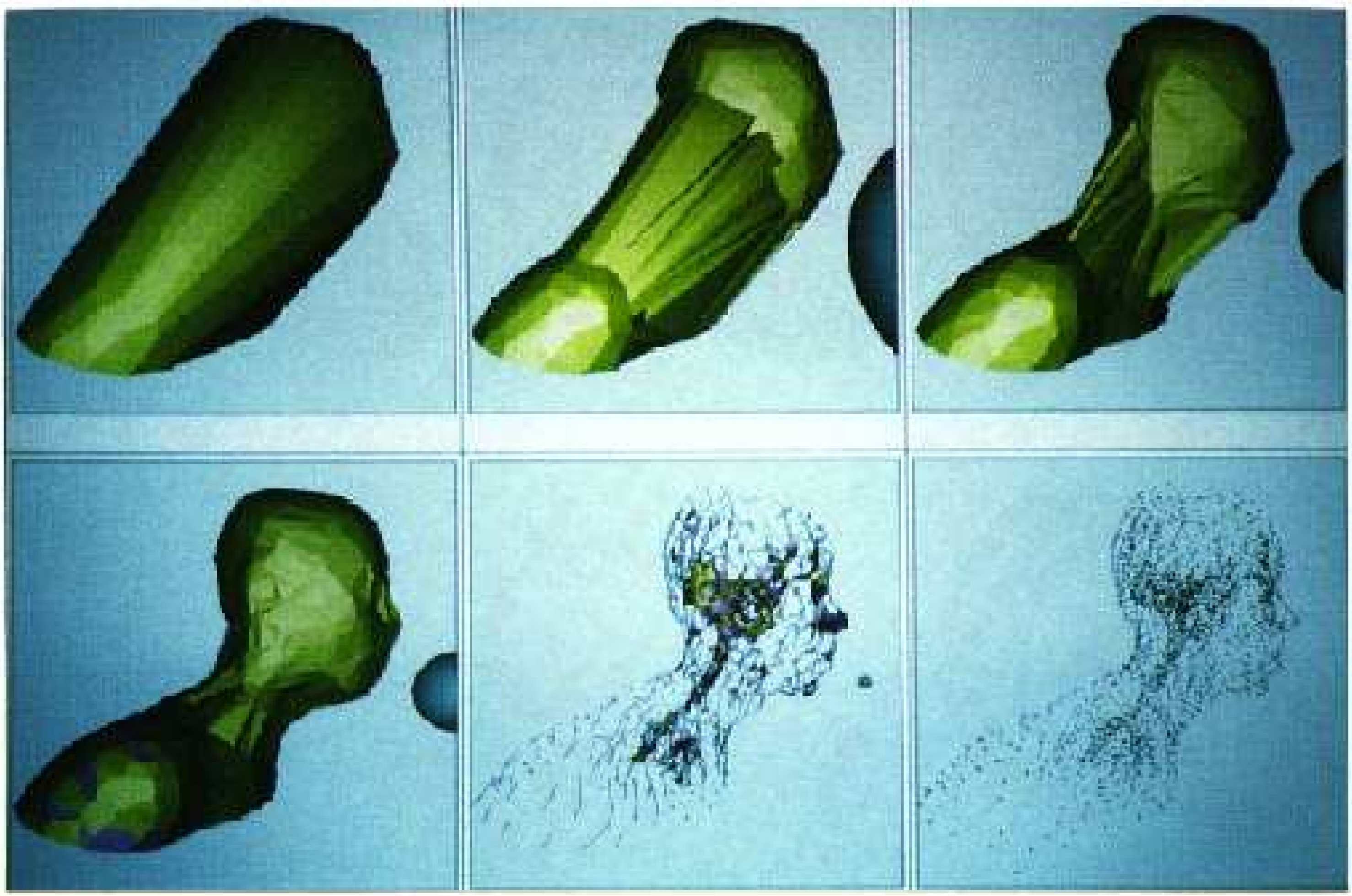


Alpha Controls the desired level of detail.

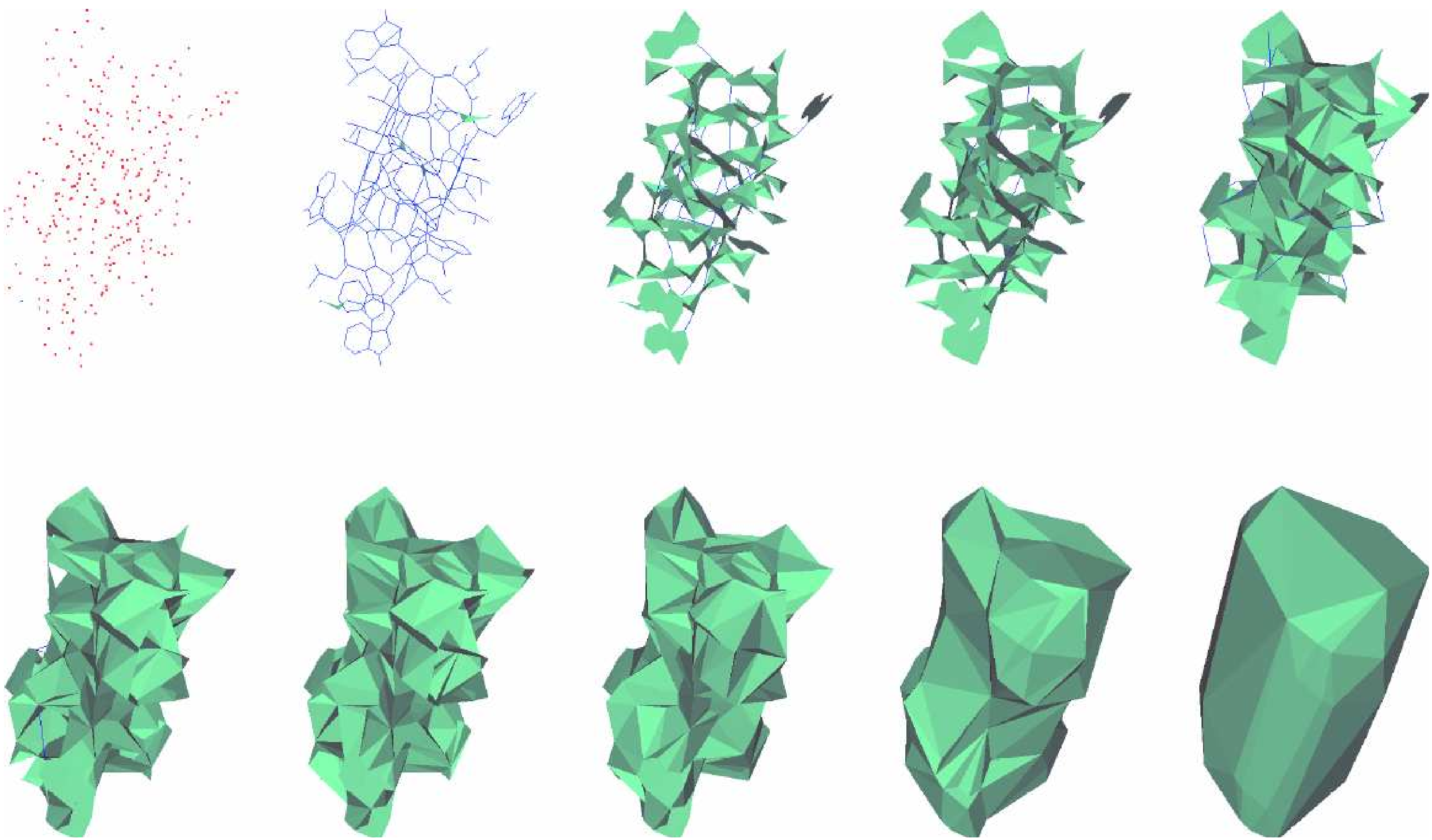


$$\alpha = \infty$$

Variable Alpha



Last set of sample alpha shapes.



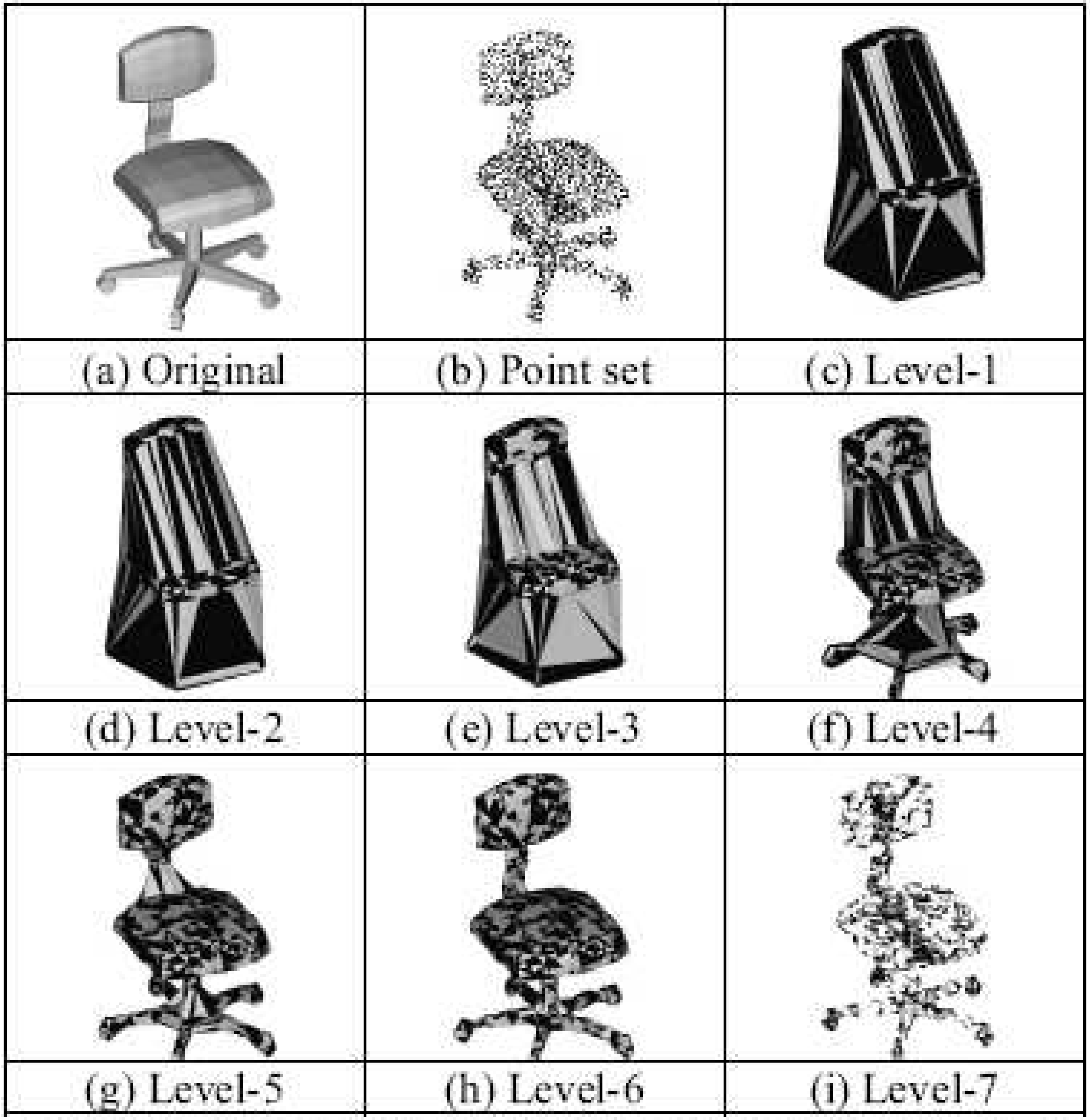


Figure 3. A set of alpha shapes generated from a set of 1024 point generated on the surface of the original model.

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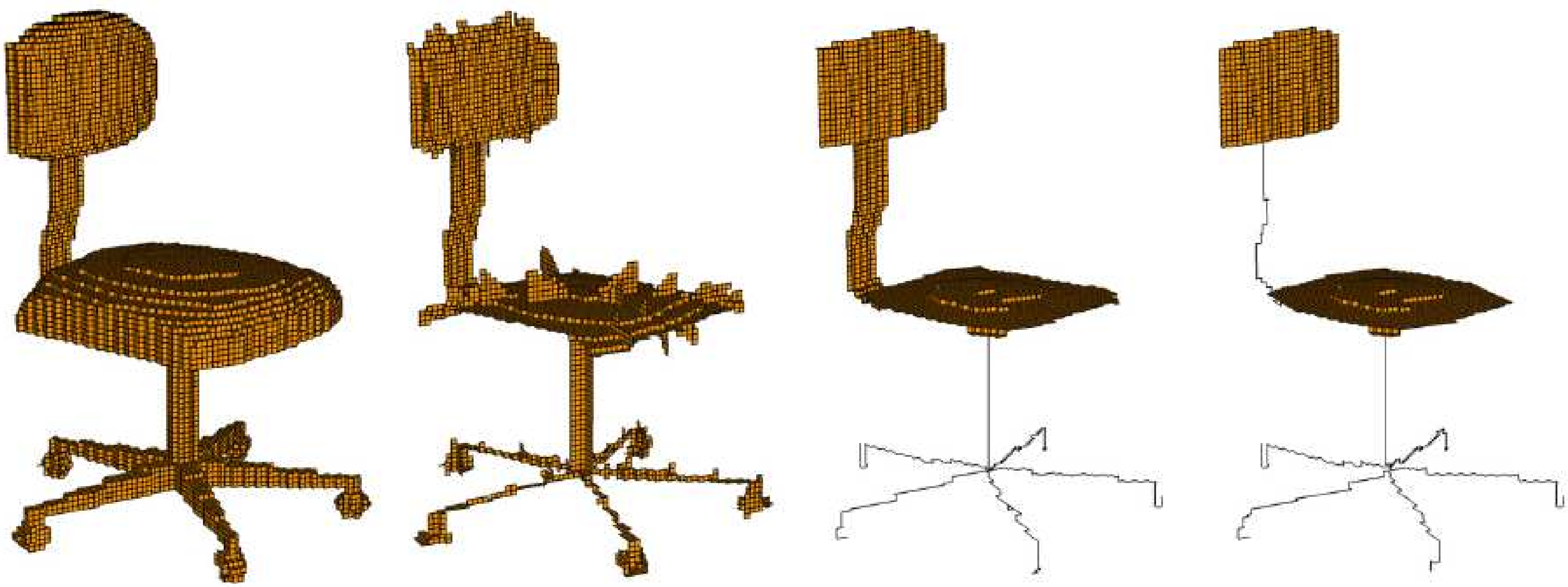
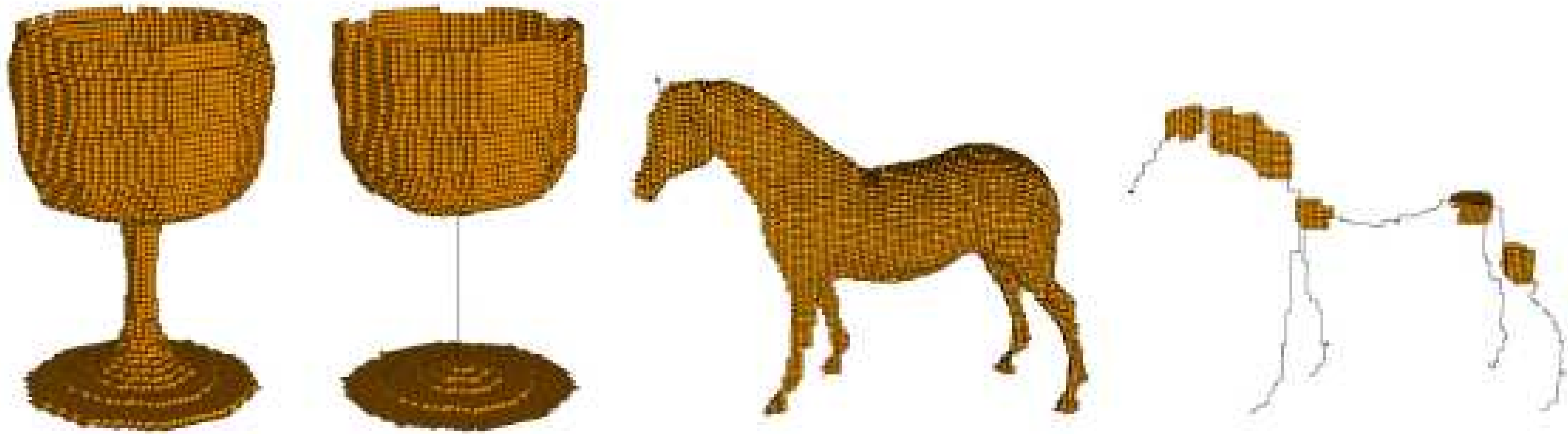
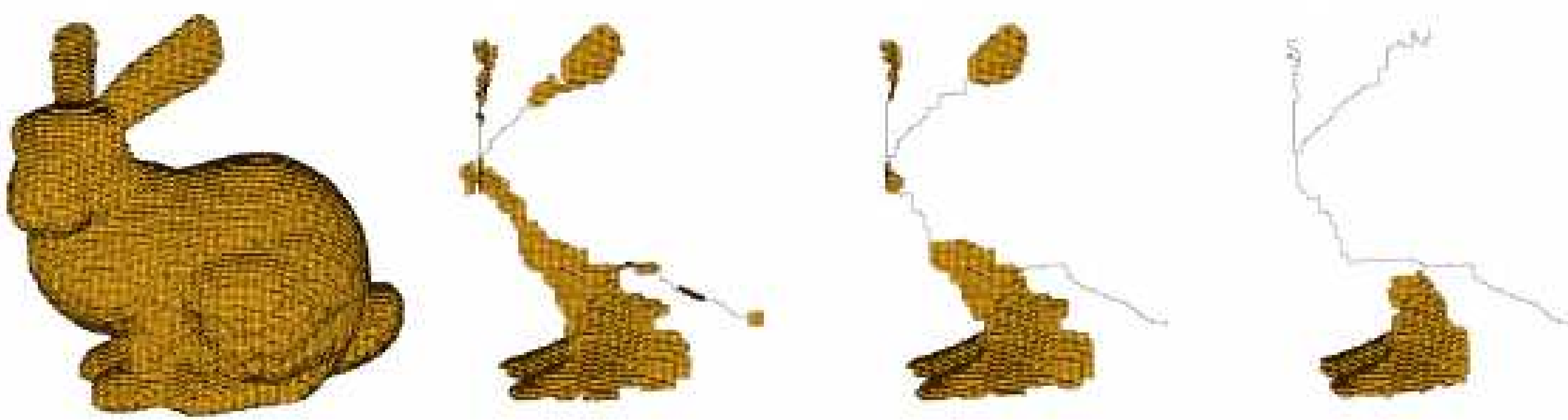


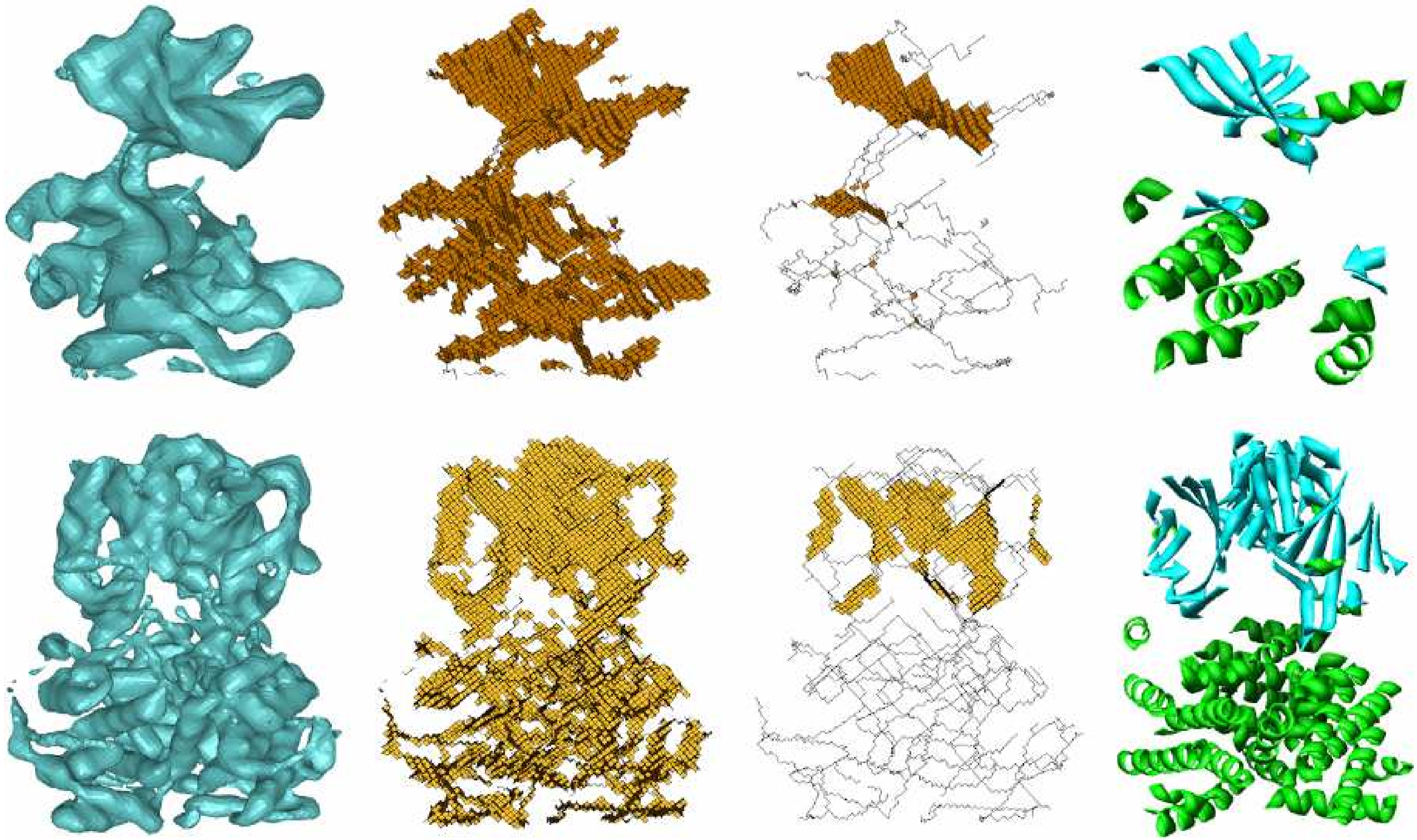


(a) (b) (c) (d)

11. A goblet (a) and a horse (c) with their skeletons (b,d). Pruning paramete
= 20 and $d_2 = 3$ are used in both examples.



(a) (b) (c) (d)



Observations

- For any α, the α-shape is a sub-graph of the Delaunay triangulation.
- The convex hull is an element of the alpha shape family.

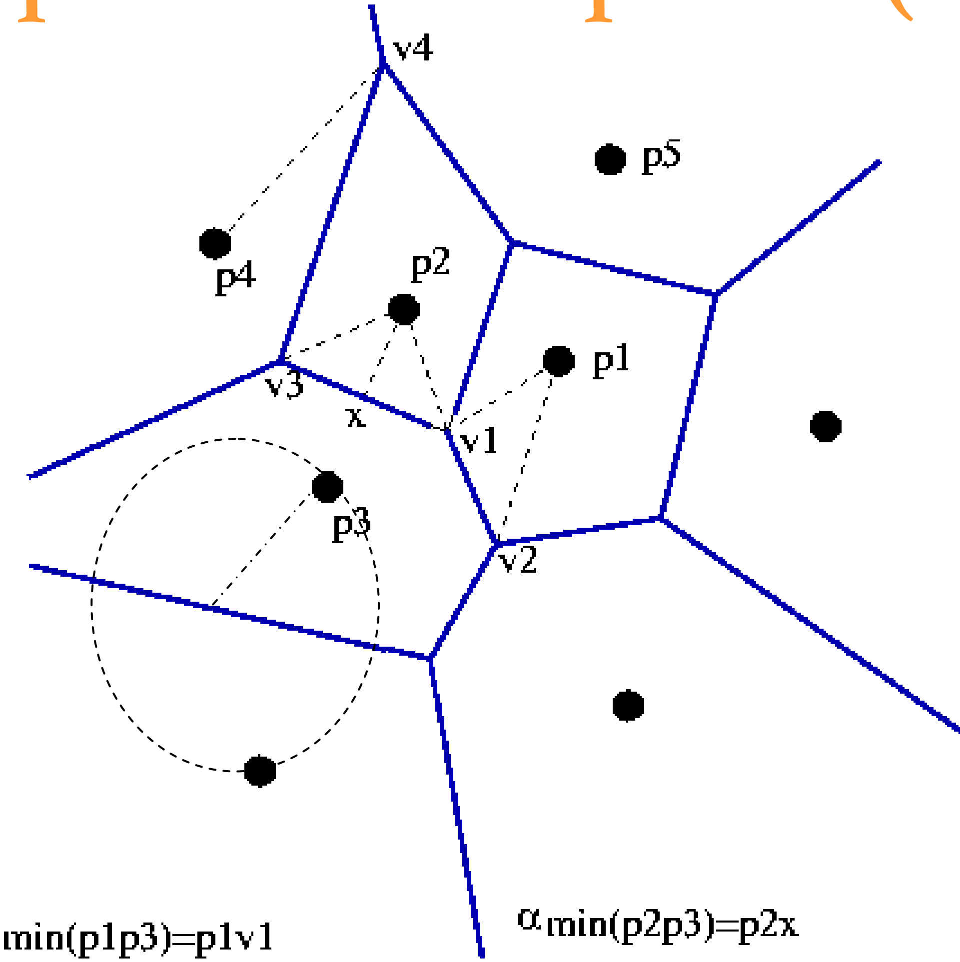
Alpha-shape of a set of points (6)

Theorem (2D case)

For each Delaunay edge $e = [p_i p_j]$ there exists $\alpha_{min}(e) > 0$ and $\alpha_{max}(e) > 0$ such that $e = [p_i p_j] \in \alpha - \text{shape}$ of S iff $\alpha_{min} \leq \alpha \leq \alpha_{max}$.

So, every alpha hull edge is in the Delaunay Triangulation, and, every Delaunay edge is in some alpha shape.

Alpha-shape of a set of points (7)



α min(p1p3)=p1v1
α max(p1p3)=p1v2

α min(p2p3)=p2x
α max(p2p3)=max{p2v1, p2v3}

α min(p4p5)=p4v4
α max(p4p5)=infinity

Alpha-shape of a set of points: algorithm(8)

- **Input**: the point set S, **output**: α-shape of S
- Compute the Voronoi diagram of S.
- For each edge e (of Voronoi or Delauney)

  compute the values $\alpha_{min}(e)$ and $\alpha_{max}(e)$.
- For each edge e

  If ($\alpha_{min}(e) <= \alpha <= \alpha_{max}(e)$) then e is in the α-shape of S.

APPLET !

Define a Voronoi Diagram:

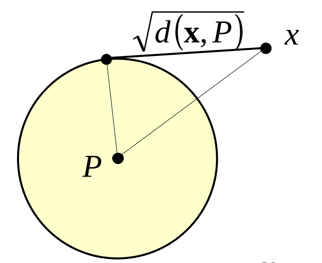
- Definition of regions:
- What have we used them for?
- What modifications might be useful?

General metrics

Generalized distance functions

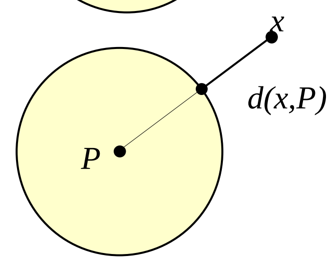
Power

$$d_p(\mathbf{x},P) = d(\mathbf{x},\mathbf{p})^2 - r_p^2$$



Additively weighted

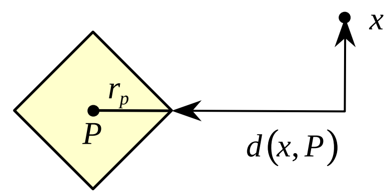
$$d_e(\mathbf{x},P) = d(\mathbf{x},\mathbf{p}) - r_p$$



Multiplicatively weighted…

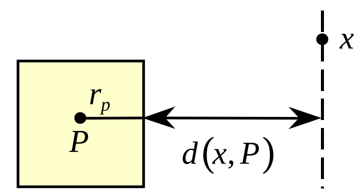
- Euclidean

$$d(\mathbf{x},P) = \sqrt{\sum_{i=1}^{d}(x_i - p_i)^2} - r_p$$



- Manhattan

$$d(\mathbf{x},P) = \sum_{i=1}^{d}|x_i - p_i| - r_p$$

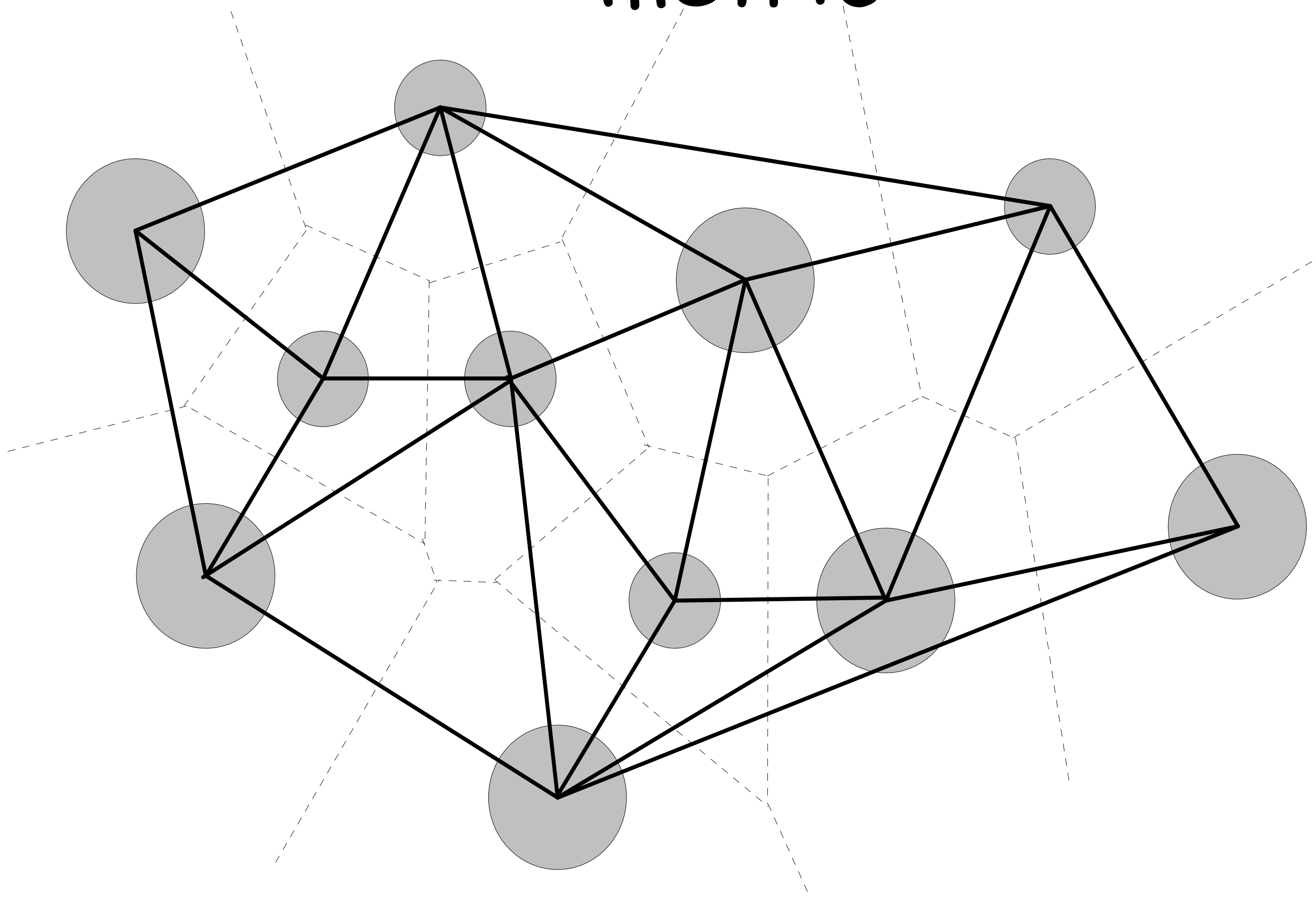


- supremum

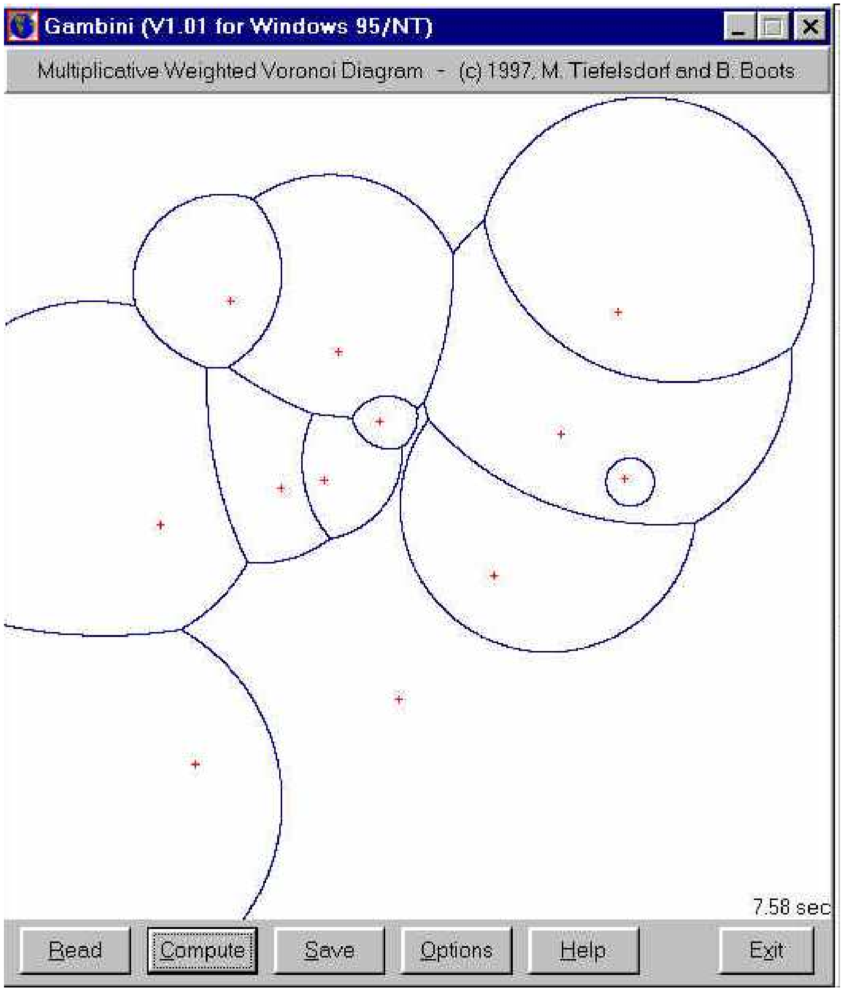
$$d(\mathbf{x},P) = \max_{i=1..d}|x_i - p_i| - r_p$$

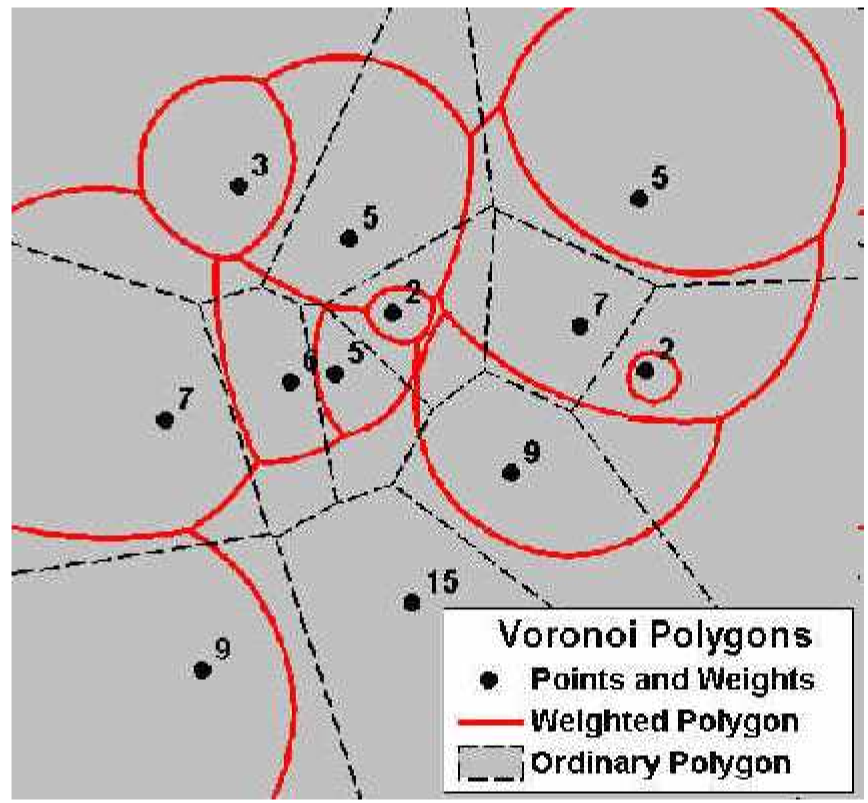
Shapes of regions?

Example: VD and DT in power metric



Multiplicatively Weighted Voronoi diagrams.





+ applet

“Walmart vs. Target” Game

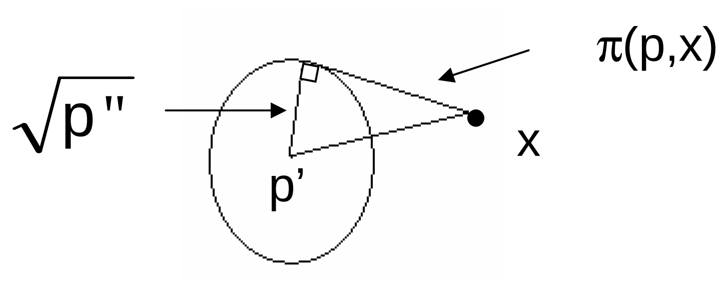
The One-Round Voronoi Game

Otfried Cheong, Sariel Har-Peled, Nathan Linial, and Jiri Matousek

In the one-round Voronoi game, the first player places n sites inside a unit-square Q. Next, the second player places n points inside Q. The payoff for a player is the total area of the Voronoi region of Q under their control. In this paper, we show that the second player can always place the points in such a way that it controls 1/2 + alpha fraction of the total area of Q, where alpha>0 is a constant independent of n.

Power diagram

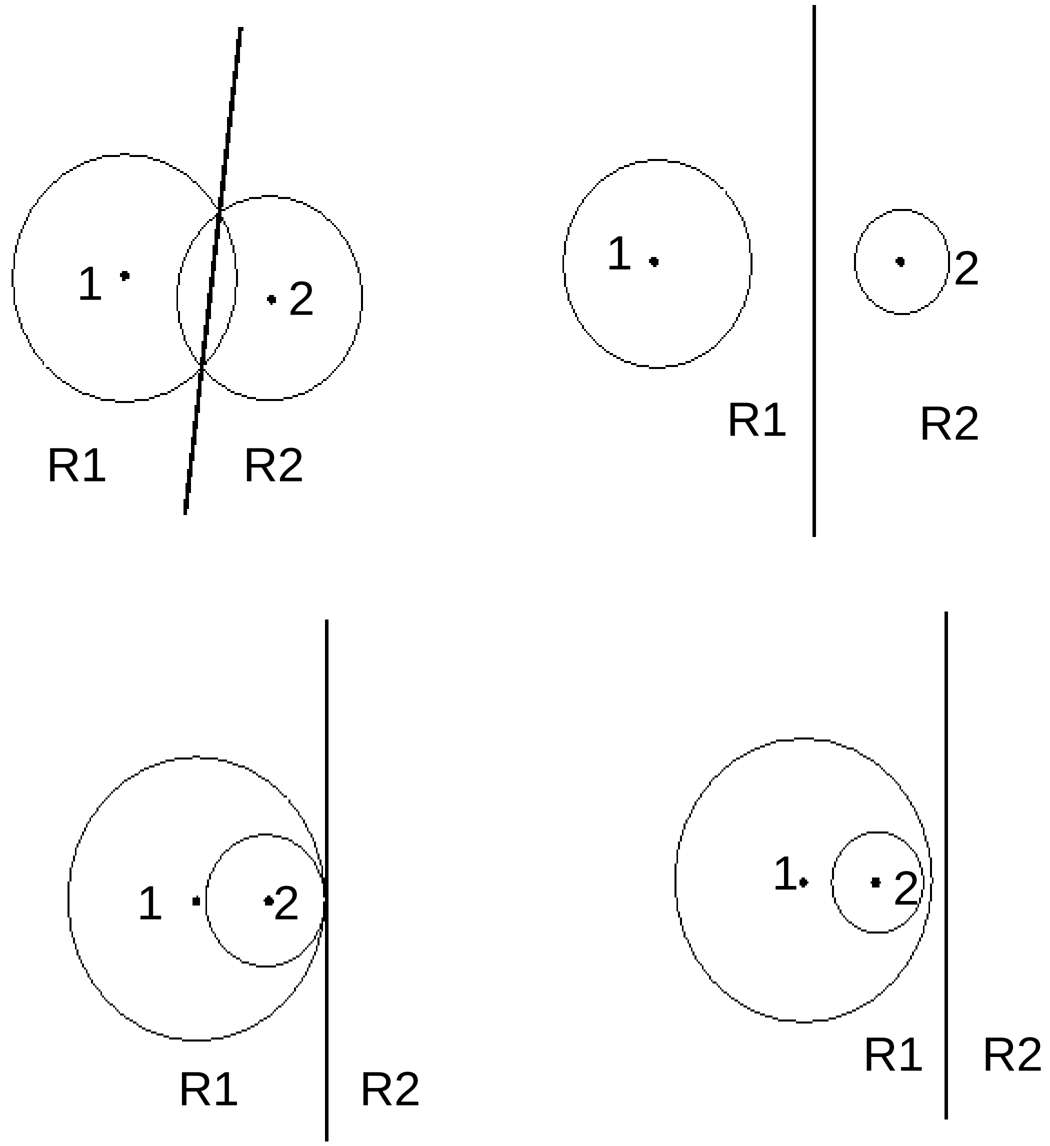
Let $S \subset R^d \times R$ be a finite set of weighted points. A weighted point is denoted as p=(p',p''), with $p' \in R^d$ its location and $p'' \in R$ its weight. For a weighted points, p=(p',p''), the power distance of a point *x* to *p* is defined as follows: $\pi(p, x) = |p' \; x|^2 - p''$



Power diagram and regular triangulation (2)

The locus of the points equidistant from two weighted points is a straight line.

Power diagram and regular triangulation (3)



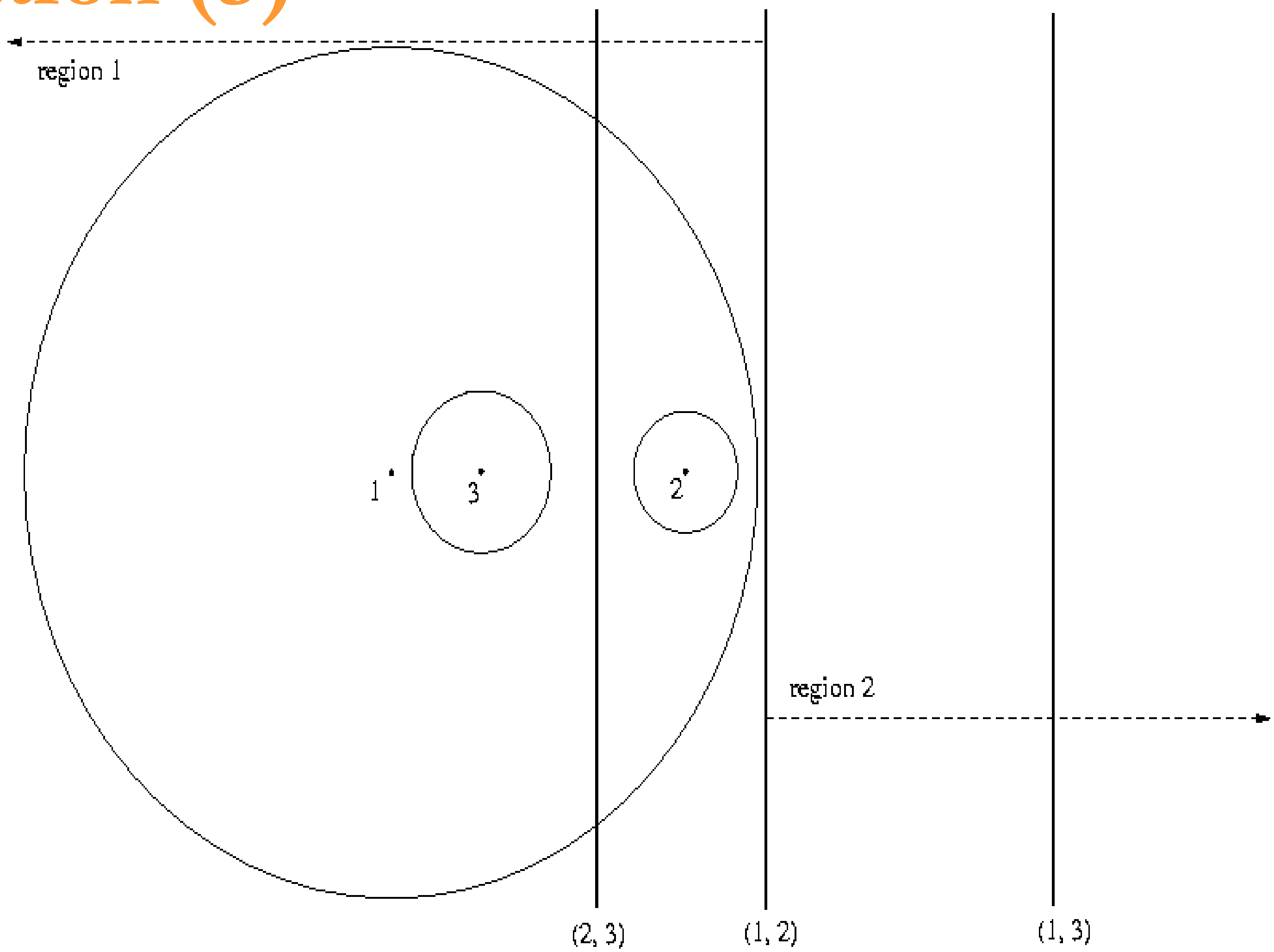
Power diagram and regular triangulation (4)

- A power region of a point p_i is defined by:

$$R(p_i) = \{x;\ \pi(p_i, x) \leq \pi(p_j, x)\ \forall j \neq i\}$$

- The power diagram of the set S, P(S), is the set of the regions $R(p_i)$.

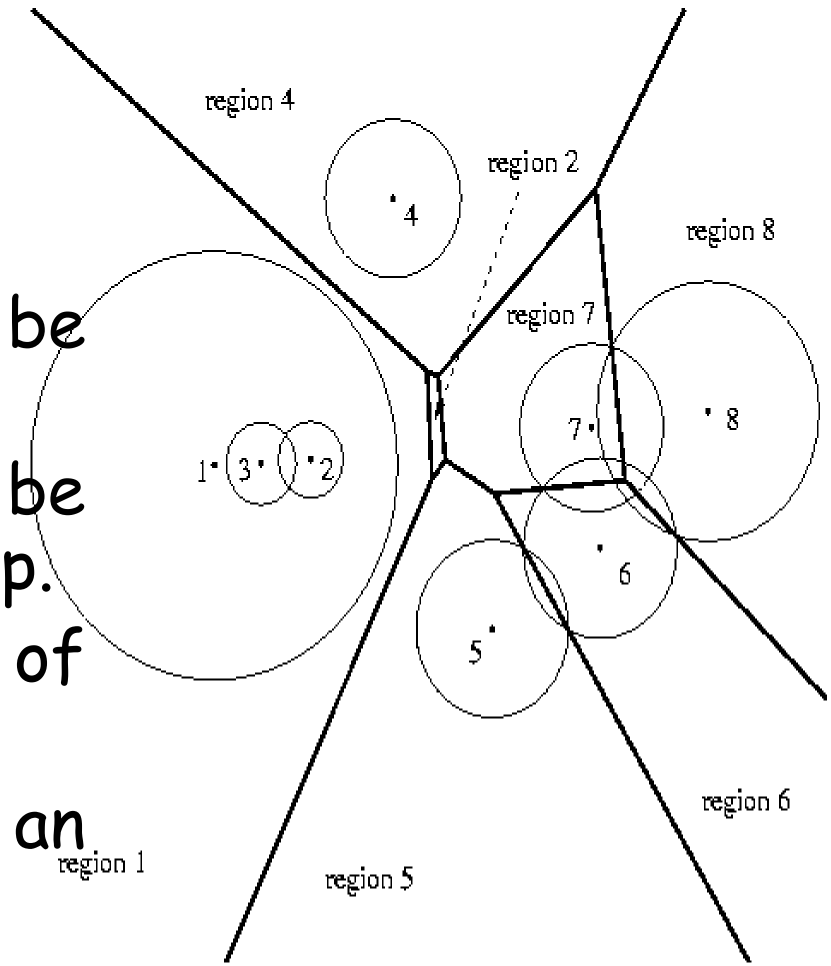
Power diagram and regular triangulation (5)



Power diagram and regular triangulation (6)

- A power region may be empty.
- A power region of p may be does not contain the point p.
- A point on the convex hull of S

  has an unbounded or an empty region.



Power diagram and regular triangulation (7)

σ_T is a k- simplex of the regular triangulation of S iff

$$\bigcap_{p \in T} R(p) \neq \phi.$$